**CALCULATING AREA OF A REGION BOUNDED BY A PARAMETRIZED CURVE**

Suppose we have a two dimensional region \( D \) to which Green’s Theorem applies. For example, take \( D \) to be a closed, bounded region whose boundary \( C \) is a simple closed \( C^1 \) curve with counter-clockwise orientation. (In general \( C \) could be a union of finitely many simple closed \( C^1 \) curves oriented so that \( D \) is on the left).

How do we calculate the area of \( D \) using line integration? In class we went through a derivation that showed that

\[ \text{Area}(D) = \frac{1}{2} \int_{C} -y \, dx + x \, dy \]

where the right hand side is a line integral over the curve \( C \) with counter-clockwise orientation. In other words, to calculate the area of the region \( D \) we must perform a line integration of the vector field \( F(x, y) = -\frac{y}{2}i + \frac{x}{2}j \) over the boundary curve.

It may seem strange that integrand on the right hand side (the vector field \( F \)) is always the same, and independent of the actual region \( D \) in this formula. Note that this is similar to your familiar notion of calculating areas via double integration:

\[ \iint_{D} 1 \, dA. \]

Here again, the integrand (which is 1) always is the same, and we change the integration domain to determine the area of the particular region we are interested in.

**1. Example: Problem 15, 6.2**

In the homework problem 15, you are asked to find the area of the closed loop of the curve

\[ x(t) = (1 - t^2, t^3 - t). \]

The first step is to figure out the range of \( t \) that corresponds to traversing around the loop. By setting \( x(t_1) = x(t_2) \) and solving for \( t_1, t_2 \) where they do not equal each other, we find that the region of interest is enclosed by the curve \( C : x(t) \) for \( t_1 = -1 \leq t \leq 1 = t_2 \). We note that this curve is oriented clockwise.

We know from above that the area of the region enclosed is given by:

\[ \frac{1}{2} \int_{-C} -y \, dx + x \, dy \]
where we have flipped the orientation of $C$ to match the assumptions of Greens theorem that we must satisfy. Evaluating this,

$$\frac{1}{2} \int_C -ydx + xdy = \frac{1}{2} \int_C -ydx + xdy$$

$$= \frac{1}{2} \int_1^{-1} (t-t^3)(-2t) + (1-t^2)(3t^2-1)dt$$

$$= \frac{8}{15}.$$ 

Here in the second to last equality we have used the parametric equation of the curve $x(t)$ to represent $x, y, dx, dy$ in terms of $t$.

2. Alternative Formulations

What was special about the vector field $\mathbf{F}(x, y) = -\frac{y}{2} \mathbf{i} + \frac{x}{2} \mathbf{j}$? Could we have used other vector fields? The answer is yes: for example, we could also use $\mathbf{F}(x, y) = -y \mathbf{i}$ or $\mathbf{F}(x, y) = x \mathbf{j}$. Can you see why? (See if you can confirm this by looking at the derivation of this formula from class notes or p. 430 in your book). In particular, you can see that in last week’s homework solutions for problem 15, $\mathbf{F}(x, y) = x \mathbf{j}$ is used, and we arrive at the same answer.