Equivalence Groups For Balance Equations and Some Applications

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If a given set of equations contains some arbitrary functions or parameters we have in fact a family of sets of equations of the same structure. Almost all field equations of classical continuum physics possess this property since they describe certain common or similar behaviors of diverse materials.

The equivalence groups are defined as groups of continuous transformations which leave a given family of equations invariant. In other words, they map an arbitrary member of the family onto another member of the same family and they transform a solution of a set of equations onto a solution of another set in the same family.

Balance equations involving arbitrary number of dependent and independent variables are given in the following form:

\[ \frac{\partial \Sigma^{\alpha i}}{\partial x^i} + \Sigma^\alpha = 0, \quad i = 1, 2, \ldots, n, \quad \alpha = 1, 2, \ldots, N \]

where \( \Sigma^{\alpha i} \) and \( \Sigma^\alpha \) are smooth functions of independent variables \( x^i \), dependent variables \( u^\alpha \) and its derivatives. A great number of filed equations of continuum physics fall into this category with \( n=4 \). In this study equivalence groups associated with first order balance equations are considered. Since, this kind of systems consist a wide class of equations of mathematical physics, their general solutions can be easily applied to most of the partial differential
equation systems by defining appropriate $\Sigma^\alpha$, $\Sigma^\beta$.

For such equations symmetry transformations $\bar{x} = \bar{x}(x, u)$, $\bar{u} = \bar{u}(x, u)$ imply

$$\frac{\partial \Sigma^\alpha(x, u)}{\partial x^i} + \Sigma^\alpha(x, u) = 0 \rightarrow \frac{\partial \Sigma^\alpha(\bar{x}, \bar{u})}{\partial \bar{x}^i} + \Sigma^\alpha(\bar{x}, \bar{u}) = 0$$

and they transform a solution $u = u(x)$ of an equation to another solution $\bar{u} = \bar{u}(x)$ of the same equation, whereas equivalence transformations $\bar{x} = \bar{x}(x, u)$, $\bar{u} = \bar{u}(x, u)$ imply

$$\frac{\partial \Sigma^\alpha(x, u)}{\partial x^i} + \Sigma^\alpha(x, u) = 0 \rightarrow \frac{\partial \Sigma^\alpha(\bar{x}, \bar{u})}{\partial \bar{x}^i} + \Sigma^\alpha(\bar{x}, \bar{u}) = 0$$

and a solution $u = u(x)$ of a given equation is transformed to a solution $\bar{u} = \bar{u}(x)$ of another equation of the same family.

In order to determine groups of equivalence transformations we employ a geometrical approach based on Cartan’s formulation of partial differential equations in terms of exterior differential forms. The determining equations of the equivalence groups and their explicit solutions are obtained for the balance equations involving arbitrary number of dependent and independent variables. And finally, the general scheme is applied to the Maxwell’s equations of the electrodynamics associated with various constitutive equations.