MATH 3283W
Quiz 2: 15 minutes to complete. This quiz is closed books, closed notes, no phones or electronic devices allowed.

Tuesday 10-3-17
Name _______________ Solutions

Remember, your work will be graded on the quality of your writing as well as on the validity of the mathematics. The quiz is worth 20 points, five of which are for a writing score.
Do not use symbols for logical connectives and quantifiers. That is, do not use the symbols $\implies$, $\iff$, $\land$, $\lor$, $\neg$, $\forall$, $\exists$, and $\exists$.
This quiz is problem 20, page 80 of your textbook. (Not in your homework set, but very much related to problems in your homework set.)

TWO PART QUESTION For both parts, suppose that $f : A \rightarrow B$ and suppose that $C \subseteq A$ and $D \subseteq B$ are subsets.

(a). (6 points) Prove or give a counterexample to the following,

$$f(C) \subseteq D \text{ if and only if } C \subseteq f^{-1}(D).$$

This statement is true.

Proof. We have to show two statements separately:

1. If $f(C) \subseteq D$, then $C \subseteq f^{-1}(D)$.
2. If $C \subseteq f^{-1}(D)$, then $f(C) \subseteq D$.

Proof of 1. Let $x \in C$. Then $f(x) \in f(C) \subseteq D$. So $f(x) \in D$. Thus $x \in f^{-1}(D)$.

Proof of 2. Let $y \in f(C)$. Then there exists $x \in C$ such that $f(x) = y$.

Since $C \subseteq f^{-1}(D)$, then $f(x) \in D$. Thus $f(x) = y \in D$, since $y$ was arbitrary, we have shown $f(C) \subseteq D$.

(b). (9 points) What condition on $f$ will ensure the following,

$$f(c) = D \text{ if and only if } C = f^{-1}(D).$$

Prove that the condition you provide here is sufficient.

Claim. If $f$ is bijective, then $f(C) = D$ if and only if $C = f^{-1}(D)$.

Proof. We have to prove two statements separately:

1. If $C = f^{-1}(D)$, then $f(C) = D$.
2. If $f(C) = D$, then $C = f^{-1}(D)$.

Proof of 1. By (a) above, it is known under the hypothesis that $f(C) \subseteq D$. So we only need to show that $D \subseteq f(C)$. Let $y \in D$. Since $C = f^{-1}(D)$ and $f$ is surjective, there exists $x \in C$ such that $f(x) = y$. But $f(x) \in f(C)$, hence $y \in f(C)$. Thus $D \subseteq f(C)$.

Proof of 2. By (a) above, it is known under the hypothesis that $C \subseteq f^{-1}(D)$. So we only need to show that $f^{-1}(D) \subseteq C$. Let $x \in f^{-1}(D)$. Then there exists $y \in D$ such that $f(x) = y$.

Since $f(C) = D$, there exists $x \in C$ such that $f(x) = y$. But then $f(x) = f(x)$. Since $f$ is injective, it follows $x = x$, and so $x \in C$. Therefore $f^{-1}(D) \subseteq C$. 