Directions:

PLEASE DO NOT OPEN EXAM UNTIL DIRECTED TO DO SO.

This is a closed book exam. No books. No notes. No crib sheets. No calculators.

You are allowed 120 minutes to complete this exam.

Please show all your work on the enclosed pages. You are not allowed any scratch paper of your own.

There are 8 questions. Including this title page, and three blank pages at the back of the exam, there are 15 pages. There are 43 possible points.

Please make sure all 15 pages are here before beginning your 120 minutes of work.

(1) (5 pts)__________
(2) (5 pts)__________
(3) (9 pts)__________
(4) (4 pts)__________
(5) (5 pts)__________
(6) (5 pts)__________
(7) (5 pts)__________
(8) (5 pts)__________
(1). (a) [2 pts] Fix a point \( x_0 \in \mathbb{R} \), and a sequence \( a_0, a_1, a_2, \ldots \) of real numbers. State what is the radius of convergence of the power series,

\[
\sum_{n=0}^{\infty} a_n (x - x_0)^n.
\]

(In other words, give an expression for the radius of convergence in terms of the numbers given above.)
(b). [3 points] Assume that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at the point $x_0$. Define what this means, and then prove that $f$ is continuous at $x_0$. 
(2). [5 pts]. Consider the following set,

\[ S = \text{The set of all finite subsets of \{1, 2, 3, \ldots\}.} \]

Is this set \( S \) countable or uncountable? Give a proof to justify your assertion.
(3). Assume \( f : \mathbb{R} \to \mathbb{R} \) is continuous, and assume also that the set \( A \subset \mathbb{R} \) is compact.

Give a proof or a counterexample for each of the following three statements:

(a). [3 pts] \( f^{-1}(A) \) is closed.

(b). [3 pts] \( f^{-1}(A) \) is compact.
(c). [3 pts] $f(A)$ is closed.
(4) [4pts] Assume $A \subset \mathbb{R}$ is compact, and assume also $x_0 \in A$.
Assume also that $\{x_n\} \subset A$ is a sequence in $A$ such that every convergent subsequence of $\{x_n\}$ converges to $x_0$.

Prove that the sequence $\{x_n\}$ converges.
5. Suppose $f \in C^2(0, \infty)$. Suppose also that we write

$$M_j \equiv \sup_{x \in (0, \infty)} |f^{(j)}(x)|,$$

where $j = 0, 1, 2$.

(a) [3 pts] Use the Taylor expansion around any fixed $x$ to show that for all $h \in (0, \infty)$,

$$|f'(x)| \leq h \cdot M_2 + \frac{M_0}{h}$$

for any $h > 0$. 
(b) [2 pts] Find the value of $h$ which minimizes the right hand side of the inequality (1) above (hint: you find this value of $h$ by just using arguments from first semester calculus! Make sure and give some reason why your value of $h$ minimizes the right hand side.)

Show how your answer implies that

$$M_1^2 \leq 4M_0 \cdot M_2.$$
(6). [5 pts] Assume that $f_n \to f$ uniformly on $\mathbb{R}$ and also assume that for some fixed $x_0 \in \mathbb{R}$, the following number exists for all $n$,

$$a_n \equiv \lim_{x \to x_0} f_n(x).$$

Prove that the limit

$$\lim_{n \to \infty} a_n$$

exists. (On a homework you did even more - you evaluated what the limit is. Here you are only asked to prove that the limit exists. Note that we do not assume continuity of the $f_n$ nor of $f$.)
(7). [5 pts.] Let \( \{c_n\} \) be any sequence of positive numbers. Prove that

\[
\liminf_{n \to \infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n \to \infty} \sqrt[n]{c_n}.
\]

Note: in your proof, you may assume without justification (if you find it convenient) that \( \lim_{n \to \infty} \sqrt[n]{p} = 1 \) for any \( p > 0 \).

Remark: The following probably won’t help you do this problem, but it gives some idea why we are interested in it. You can also show that \( \limsup_{n \to \infty} \sqrt[n]{c_n} \leq \limsup_{n \to \infty} \frac{c_{n+1}}{c_n} \). Taken together, this inequality and the previous show that: if the ratio test establishes convergence, so too the root test. And if the root test is inconclusive, so too the ratio test. In other words, the root test is a stronger test for convergence of series than the ratio test.
(8). [5 points] Assume that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following two conditions,

(a). For each compact set $K \subset \mathbb{R}$, the set $f(K)$ is compact.

(b). For any nested decreasing sequence of compact sets $K_1 \supseteq K_2 \supseteq K_3 \cdots$, we have

$$f(\cap_{n=1}^{\infty} K_n) = \cap_{n=1}^{\infty} f(K_n).$$

Prove that $f$ is continuous.
this is scratch paper
this is scratch paper
this is scratch paper