The Way of Analysis
p. 231:

2). Prove the linearity of the Riemann integral.
If \( f, g \) are both integrable on \([a, b]\), then we evaluate their integrals by picking a single sequence of partitions \( P_k \) for which \( \text{mesh}(P_k) \) goes to zero, and some Riemann sums \( S(f, P_j), S(g, P_j) \) which are then guaranteed (by the big long theorem we proved) to converge to the integrals of \( f, g \) respectively. But then \( S(f + g, P_j) \) converges to the integral of \( f + g \), and the number that it converges to will be the sum of the two integrals, since the process of taking limits is linear. Similarly, you show \( \int_a^b c f \, dx = c \int_a^b f \, dx \) for any constant \( c \in \mathbb{R} \).

3.) Prove that \( \int_a^b f + \int_c^b f = \int_c^a f \):
Let \( P_j \) and \( Q_j \) be partitions of \([a, b]\) and \([b, c]\) with size (i.e. “mesh”) going to zero. Then \( R_j = P_j \cup Q_j \) is a sequence of partitions of \([a, c]\) with size going to zero and
\[
S(f, R_j) = S(f, P_j) + S(f, Q_j).
\]
Taking the limit as \( j \) goes to \( \infty \) gives the claim (note as in the previous problem that theorem 6.2.1 allows us to check things for just one sequence of partitions).

6.) Suppose that \( f \) is Riem. Int. on \([a, b]\) and \( f(x) = g(x) \) except at a finite number of points. Show that \( g \) is Riem. Int.:
Just use the same proof as for theorem 6.2.3 (i.e., surround the finite number of points by small intervals so the oscillation is small there; then use the Riem. Int. of \( f \) to choose a partition of what’s left over so the oscillation is small there too).

9.) Suppose that \( f \) is Riem. Int. and bounded on \([a, b]\) and define
\[
F(x) = \int_a^x f(t) \, dt.
\]
Show that \( F \) is continuous:
If \( y < x \), then the additivity of the integral gives
\[
F(x) - F(y) = \int_y^x f(t) \, dt = \int_y^x f(t) \, dt = \int_y^x f(t) \, dt \leq |x - y| \sup |f|.
\]
This shows that \( F \) is Lipschitz and therefore cts.

10.) Solution to be added later.

11.) Suppose that \( f \) is cts on \([a, b]\) and diff. on \((a, b)\) where \( f' \) is Riem. Int. Show that
\[
\int_a^b f'(x) \, dx = f(b) - f(a);
\]
Suppose that \( P \) is a partition. By the MVT, for each interval \([x_{j-1}, x_j]\) there is a point \( y_j \) with
\[
f(x_j) - f(x_{j-1}) = (x_j - x_{j-1}) f'(y_j).
\]
In particular, the Cauchy sum is
\[ S(f', P) = \sum [(x_j - x_{j-1}) f'(y_j)] = \sum [f(x_j) - f(x_{j-1})] = f(b) - f(a). \]
Taking a sequence of partitions with size going to zero and taking the limit of the Cauchy sums gives the claim.

12). This follows immediately from DeMorgan’s laws.

\textbf{p. 235:}

1.) For which \(a\) and \(b\) does
\[ \int_0^{1/2} x^a |\log x|^b \, dx \]
exist?

Sketch of solution: This exists for every \(a > -1\) and does not exist for any \(a < -1\) (by the same argument as on page 233).

When \(a = -1\), it exists only for \(b < -1\).

3.) Suppose that \(f \geq 0\) and \(\int_1^\infty f\) exists. Do we have to have \(\lim_{x \to \infty} f(x) = 0\)? Must \(f\) be bounded? What can we say if \(\lim_{x \to \infty} f(x)\) exists?

The answer to the first two questions is NO! For example, suppose that
\[ f(x) = \sum_{j=1}^{\infty} \left[j \chi_{[j,j+2^{-j}/j]}\right]. \]
This function is integrable because \(\sum [j (2^{-j}/j)] = \sum 2^{-j} = 1\). On the other hand, this \(f\) is not bounded and \(\lim_{x \to \infty} f(x)\) does not exist.

On the other hand, if the limit does exist then it must be zero. Otherwise, we would get that for all \(x > N\) we have \(f(x) > \epsilon > 0\). Since \(f \geq 0\), this would imply
\[ \int_1^M f > \int_N^M \epsilon = \epsilon (M - N) \]
which goes to infinity as \(M\) goes to infinity.

5). Solution to be added later