HW 3 SOLUTIONS

The Way of Analysis
p. 54:

2.) Prove that every real number $x$ has a unique cube root. This has two parts: show $x$ has at least one cube root and then show that $y^3 = x$ and $z^3 = x$ implies that $y = z$. The first part just uses divide and conquer exactly as for square roots (pages 53–54).

Suppose now that $y^3 = z^3$ and $y, z \neq 0$ (it is easy to see that only 0 solves $y^3 = 0$). We can factor this to get

$$(y - z)(y^2 + yz + z^2) = 0.$$ \[1\]

Since

$$y^2 + yz + z^2 = (y + z)^2/2 + y^2/2 + z^2/2 \geq y^2/2 + z^2/2 > 0,$$
we see that $(y - z) = 0$ as desired.

3.) Suppose that $|x_n| \leq 2^{-n}$ and $y_n = \sum_{i=1}^{n} x_i$. Prove that $y_n$ converges. Note that

$$\sum_{i=j}^{N} |x_n| \leq \sum_{i=j}^{N} 2^{-j} = 2^{1-j} - 2^{-N} < 2^{1-j}.$$ \[2\]

It follows that for $k > j > M$ we have (by the triangle inequality)

$$|y_k - y_j| = \left| \sum_{n=j+1}^{k} x_n \right| \leq \sum_{n=j+1}^{k} |x_n| < 2^{-M}.$$ \[3\]

Since this can be made arbitrarily small for $M$ sufficiently large, we see that $y_n$ is Cauchy and hence converges.

7.) Prove that $a > b > 0$ implies $\sqrt{a} > \sqrt{b}$.

Using the arithmetic rules for a field, we see that

$$a - b = (\sqrt{a} + \sqrt{b}) (\sqrt{a} - \sqrt{b}).$$ \[4\]

Since $\sqrt{a}$ and $\sqrt{b}$ are positive (by definition; see Theorem 2.3.3), so is $(\sqrt{a} + \sqrt{b})$. Therefore, $(\sqrt{a} + \sqrt{b})$ has a positive inverse $1/(\sqrt{a} + \sqrt{b})$. We can now write $(\sqrt{a} - \sqrt{b})$ as the product of two positive numbers:

$$\sqrt{a} - \sqrt{b} = (a - b) \left[1/(\sqrt{a} + \sqrt{b})\right].$$ \[5\]

10.) Prove that the irrationals $\mathbf{Q}^c$ are dense in $\mathbf{R}$; i.e., any $x \in \mathbf{R}$ can be represented by a Cauchy sequence of irrationals. Note that if $y \in \mathbf{Q}^c$ and $z \in \mathbf{Q}$, then $yz \in \mathbf{Q}^c$ (since $y = [z(yz)^{-1}]^{-1}$). Since $\sqrt{2} \in \mathbf{Q}^c$, it follows that $\sqrt{2}/N \in \mathbf{Q}^c$ for every $N \in \mathbf{N}$. Namely, we get arbitrarily small irrational numbers.

Suppose now that $x_j$ is a Cauchy sequence of rationals for $x$. You should now try and show that the sequence of irrationals

$$y_j = x_j + \sqrt{2}/j$$
then converges to $x$. (This is relatively straightforward, but should be included in your solution.)

**p. 84:**

1.) Find the limit points:
   - a) $1/n + (-1)^n$ has limsup $1$, liminf $-1$, and no other limit points (since $1/n$ goes to zero).
   - b) $1 + (-1)^n/n$ has limit 1. (Hence this is the only limit point.)
   - c) $(-1)^n + 1/n + 2 \sin(n\pi/2)$ has limit points 1 and $-3$. To see this, we can throw away the $1/n$ since this goes to zero; then notice that $2 \sin(n\pi/2)$ is just
     $$2, 0, -2, 0, 2, 0, -2, 0, \ldots$$

2.) Suppose $x_n = y_n + z_n$ is bounded where $y_n$ increasing and $z_n$ decreasing. Does it converge? Maybe not! For example, if $y_n = n$ and $z_n$ is the sequence
   $$-1, -1, -3, -3, -5, -5, \ldots,$$
   then $y_n + z_n$ is the sequence
   $$0, 1, 0, 1, 0, 1, \ldots,$$
   However, if $y_n$ and $z_n$ are bounded, then they converge to real numbers (by our theorem about monotone sequences) and hence so does $x_n$ (since the sum of convergent sequences converges).

4.) Prove $\sup(A \cup B) \geq \sup A$ and $\sup(A \cap B) \leq \sup A$. By definition, if $y = \sup(A \cup B)$, then $y \geq a$ for any $a \in A \cup B$. In particular, $y$ is an upper bound for $A$. Since $\sup A$ is the least upper bound, we get
   $$\sup(A \cup B) \geq \sup A .$$
   The second claim follows from the first by writing $A = (A \cap B) \cup A$.

7.) Construct a sequence whose limit points are exactly the integers. To simplify notation, we will do it for the natural numbers and leave the generalization to the integers for you to work out. Here is the sequence:
   $$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots$$
   First of all, this sequence contains every natural number infinitely often so automatically has every natural number as a limit point. Second, since any two natural numbers differ by distance at least one, any limit point must also be a natural number.

11.) Let $a_{ij}$ be a rectangular array and consider the diagonal sequence
   (D) $a_{11}, a_{21}, a_{31}, a_{22}, a_{13}, \ldots$
   Prove that any limit point of a row or column is a limit point of this sequence.
   The cases of rows versus columns are symmetric, so we will just deal with one. Suppose therefore that the $i$-th row has a convergent subsequence
   (R) $a_{ij_1}, a_{ij_2}, a_{ij_3}, \ldots$
   with limit $a$. Note that the sequence $j_n$ is increasing and, therefore, so is the sum of the indices $i + j_n$. This means that $a_{ij_{n+1}}$ occurs after $a_{ij_n}$ in (D) and therefore (R) is also a subsequence of (D).
Do you necessarily get all limit points of (D) as limit points of a row or column? No! Suppose that we had an array where $a_{ij} = \delta_{ij}$, i.e., where $a_{ij}$ is one when $i = j$ and zero otherwise. In this case, the $k$-th row or column is zero after the $k$-th term, so $0$ is the only limit point of a row or column. However, infinitely many ones appear in (D), so $1$ is a limit point of (D).