HW 6 SOLUTIONS

The Way of Analysis
p. 152:
1.) The question is equivalent to asking: show that $O(h^2)$ implies $o(h)$ as $h \to 0$:
By definition, $f \in O(h^2)$ implies that there exist $C$ and $\delta$ so that for $|h| < \delta$ we have
$$|f(h)| \leq C h^2.$$  
But then, we easily have
$$\lim_{h \to 0} \frac{|f(h)|}{h} \leq \lim_{h \to 0} \frac{C h^2}{h} = 0.$$
An example which shows that the converse doesn’t hold is
$$f(x) = |x|^3.$$  
(You can easily check that the converse doesn’t hold for this function.)
5.) Show that $f(x_0) = 0$ and $f(x-x_0) = o(x-x_0)$ implies that $f'(x_0)$ exists:
The difference quotients are
$$\frac{f(x) - f(x_0)}{x-x_0} = \frac{f(x)}{x-x_0}.$$  
By the definition of the notation $o(x-x_0)$, $x \to x_0$, the limit as $x \to x_0$ of the above quotients exists and is zero. Hence, $f'(x_0) = 0$.
6.) Show that the one-sided difference quotients at 0 do not exist for $f(x) = x \sin(1/x)$ (and hence the derivative $f'(0)$ does not exist):
The function $f$ is an even function, i.e., $f(-x) = f(x)$, so we may assume that $x > 0$. Note first that $x_k = 1/(2 k \pi)$ gives a sequence of points with $f(x_k) = f(0) = 0$. In other words, this gives one sequence of difference quotients converging to 0.
On the other hand, the points $y_k = 1/(2 k \pi + \pi/2)$ gives a sequence of points with $f(y_k) = 1/(2 k \pi + \pi/2)$ and difference quotients
$$\frac{f(y_k) - f(0)}{y_k} = \frac{1/(2 k \pi + \pi/2)}{1/(2 k \pi + \pi/2)} = 1.$$  
Therefore, the limit does not exist.
7.) Give an example of a diff. function whose tangent line at a point does not stay on one side of the graph, even in any nbhd of the point:
Let $f(x) = x^3$. The tangent line at 0 is just the $x$-axis since $f(0) = f'(0) = 0$. However, every nbhd of 0 contains both positive and negative values of $f$.  