HOMEWORK 7 SOLUTIONS

The Way of Analysis
p. 163:

1.) Suppose \( f, g \) are diff. on \((a, b)\) and \( g(a) \neq g(b) \). Show there exists \( x_0 \) between \( a \) and \( b \) with

\[
\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}.
\]

Following the hint, we define,

\[
h(x) = (f(b) - f(a)) g(x) - (g(b) - g(a)) f(x),
\]

so that \( h(b) = h(a) \). The MVT gives \( x_0 \) with

\[
0 = h(b) - h(a) = (b - a) h'(x_0) = (b - a) \left[ (f(b) - f(a)) g'(x_0) - (g(b) - g(a)) f'(x_0) \right].
\]

The result follows since \( b \neq a \).

2.) Suppose that \( f \) satisfies

\[
|f(x) - f(y)| \leq M |x - y|^\alpha,
\]

for some \( \alpha > 1 \). Show that \( f \) is constant:

For any \( x, y \), we conclude immediately that \( f(y) = f(x) + 0 \cdot (y - x) + o(|x - y|) \). By the properties of the derivative (see the discussion on “unique best affine approximation”) we know \( f'(x) = 0 \) for all \( x \in R \). But now by the mean value theorem we conclude that \( f \) must be a constant.

8.) Suppose \( f \) is in \( C^1(a, b) \) and \([c, d] \subset (a, b)\). Show that given \( \epsilon > 0 \), there exists \( \delta > 0 \) so that

\[
|x - y| < \delta \implies |f(x) - f(y) - f'(y)(x - y)| \leq \epsilon |x - y|.
\]

Since \( f' \) is cts and \([c, d] \) is compact, \( f' \) is uniformly cts on \([c, d] \). Hence, there exists \( \delta > 0 \) so that if \( |y - z| < \delta \), then

\[
|f'(y) - f'(z)| < \epsilon.
\]

By the MVT, if \( |x - y| < \delta \), then we get \( z \) between \( x \) and \( y \) with

\[
|f(x) - f(y) - f'(z)(x - y)| = 0.
\]

The claim now follows from the triangle inequality:

\[
|f(x) - f(y) - f'(y)(x - y)| \leq |f(x) - f(y) - f'(z)(x - y)| + |f'(y) - f'(z)||x - y| \leq \epsilon |x - y|.
\]

10.) If \( f \) has a max at an endpoint, what can you say about the one-sided derivative (if it exists):

At the left-hand side, \( f' \leq 0 \) at a max (otherwise moving to the right increases \( f \)). Likewise, at the right-hand side, \( f' \geq 0 \).

We get the opposite inequality for mins (just by multiplying by \(-1\)).

11.) Suppose that \( f' \) is constant; show that \( f \) is affine:
Assume that $f$ is defined on all of $\mathbb{R}$ for simplicity and $f'(x) = C$. Given any $x \in \mathbb{R}$, the MVT gives some $z$ so that

$$f(x) - f(0) = x f'(z).$$

Since $f'$ is constant, this means that $f(x) = f(0) + C x$, which is what we wanted.

The same argument can be easily modified in case $f$ is defined only on some interval.