The Way of Analysis
p. 217:

1.) Suppose that \( f(x) = \int_{a(x)}^{b(x)} g(t) \, dt \) where \( a(x), b(x) \) are \( C^1 \) functions and \( g(t) \) is cts. Prove that

\[
    f'(x) = b'(x) g(b(x)) - a'(x) g(a(x)) .
\]

Let’s start by considering a modification of the problem. Suppose that \( f(x) = \int_{0}^{b(x)} g(t) \, dt \) where \( b(x) \) is a \( C^1 \) function and \( g(t) \) is cts. We want to prove that

\[
    f'(x) = b'(x) g(b(x)) .
\]

Set \( G(s) = \int_{0}^{s} g(t) \, dt \) (we can do this since \( g \) is cts). Then \( f(x) = G(b(x)) \). By the chain rule,

\[
    f'(x) = G'(b(x)) b'(x) = g(b(x)) b'(x) ,
\]

where the second equality used the fundamental theorem of calculus.

The general problem now follows by splitting up the integral into two pieces

\[
    \int_{a(x)}^{b(x)} g(t) \, dt = \int_{0}^{b(x)} g(t) \, dt + \int_{a(x)}^{0} g(t) \, dt
\]

(We are assuming here that the interval of differentiability and continuity of the functions is all of the real numbers. We need not assume though that 0 is between \( a(x) \) and \( b(x) \).

2.) Show that \( |f| \leq f|f| \):

For any partition, we obviously have \( |S(f, P)| \leq S(|f|, P) \) for any Riemann sum. Now take the limit as the maximum size of the intervals in \( P \) goes to zero.

4.) Suppose \( f \) is cts on \([a, b]\). Show there exists \( y \) so that

\[
    (b - a)^{-1} \int_{a}^{b} f = f(y) .
\]

Let \( m \) and \( M \) be the min and max of \( f \) on \([a, b]\) so that

\[
    m \leq (b - a)^{-1} \int_{a}^{b} f \leq M .
\]

The claim now follows by the IMV theorem since \( f \) is cts: the number \( (b - a)^{-1} \int_{a}^{b} f \) is between \( m \) and \( M \), and since \( f \) is continuous and takes on the values \( m \) and \( M \) at some points in \([a, b]\), it must also take on all values in between, including this particular value that we have pointed out!

5.) Let \( g \) be cts on \([a, b]\) and define \( f \) by

\[
    f(x) = \int_{a}^{x} (x - t) g(t) \, dt .
\]

Prove that \( f \) satisfies \( f'' = g \) and \( f(a) = f'(a) = 0 \).

Obviously \( f(a) = 0 \).
Thm 6.1.7 (page 213) shows how to differentiate things like \( f \); namely, if \( h(x,t) \) is a nice function of two variables, then
\[
\frac{d}{dx} \left[ \int_a^x h(x,t) \, dt \right] = h(x,x) + \int_a^x \frac{\partial h}{\partial x}(x,t) \, dt.
\]
Applying this with \( h(x,t) = (x-t)g(t) \) (and \( \frac{\partial h}{\partial x} = g(t) \)), we differentiate \( f \):
\[
f'(x) = (x-x)g(x) + \int_a^x g(t) \, dt = \int_a^x g(t) \, dt.
\]
Obviously \( f'(a) = 0 \). The fundamental theorem of calculus then gives \( f''(x) = g(x) \) as desired.

8.) Let \( f \) be \( C^1 \) on \( \mathbb{R} \) and define a function \( g \) by
\[
g(x) = \int_0^1 y^2 f(xy) \, dy.
\]
Prove that \( g \) is \( C^1 \) and find a formula for \( g' \):
By the change of variables formula (Thm 6.1.5, page 211), the substitution \( z = xy \) can be used for \( x \neq 0 \) to give,
\[
g(x) = \int_0^1 y^2 f(xy) \, dy = \frac{1}{x^3} \int_0^x z^2 f(z) \, dz.
\]
The Leibniz rule and the FTC now give that for \( x \neq 0 \),
\[
g'(x) = x^{-3} x^2 f(x) - 3 x^{-4} \int_0^x z^2 f(z) \, dz = x^{-1} f(x) - 3 x^{-4} \int_0^x z^2 f(z) \, dz.
\]
At \( x = 0 \), we use the Theorem on page 213 of the text, and the original form of the function \( g(x) \) to compute that
\[
g'(0) = \int_0^1 f'(0)y^2 \, dy
\]
which simplifies to \( g'(0) = \frac{1}{4} f'(0) \). (So speaking carefully, we should say that we cannot express \( g'(x) \) in terms of \( f \) at all values of \( x \) since at \( x = 0 \) we need to know the value of \( f'(0) \).)

12.) Suppose \( f \) is cts and periodic with \( f(x+a) = f(x) \) for some fixed \( a \). Prove that \( F(x) = \int_0^x f \) is periodic iff \( \int_0^a f(x) = 0 \):
Note that \( F(0) = 0 \) and \( F(a) = \int_0^a f \), so \( F \) periodic of period \( a \) does imply that
\[\int_0^a f(x) = 0.\]
Conversely, \( F(x+a) = F(x) = \int_0^{x+a} f - \int_0^x f = \int_x^{x+a} f. \) Since \( f \) is periodic, this is equal to \( \int_0^a f. \)

14). (a). (see text for the statement of the problem. Our solution follow’s the hint.)
Fix \( x \in [a,b] \). By the mean value theorem, \( \exists y \in (a,x) \) so that \( f(x) = f'(y)(x-a) \). It just remains to get a good bound for \( f'(y) \).
Note that \( f(a) = f(b) = 0 \), so by the mean value theorem (or Rolle’s theorem in this case.) \( \exists z \in (a,b) \) so that \( f'(z) = 0 \). I want to try and estimate \( f'(y) \) by ‘integrating’ \( f''(x) \) starting at the place where the derivative vanishes, that is, starting at \( z \).
\[
f'(y) = \int_z^y f''(t) \, dt.
\]
Hence $|f'(y)| \leq M_2|y - z|$. Now notice that if $|y - z| \leq |x - b|$, we are done. Otherwise, we must have $|y - z| > |x - b|$. This clearly implies $|a - z| > |z - b|$ (it helps if you are drawing a picture to keep track of all this...) in which case we certainly also have

$$\left| \frac{f(z) - f(a)}{z - a} \right| \leq \left| \frac{f(z) - f(b)}{z - b} \right|.$$ 

But by the mean value theorem, there must be a point $p$ in $[z, b]$ so that $f'(p)$ is equal to the right hand side of the last inequality above. But then by the intermediate value theorem (applied to the function $f'$), there is a point $\tilde{y} \in (z, p)$ with $|f'(\tilde{y})| = |f'(y)|$. Now we are basically done: you bound $f'(\tilde{y})$ by integrating $f''$ between $z$ and $\tilde{y}$. That is, we have

$$|f'(\tilde{y})| \leq \left| \int_\tilde{y}^z f''(t)dt \right| \leq M_2|\tilde{y} - z| \leq M_2|z - b| \leq M_2|x - b|.$$ 

We leave the reader to adapt this argument to the remaining case when $x > z$.

(b). (see text for statement)

Subdivide the interval $[a, b]$ with a partition wherein $x_k - x_{k-1}$ is constant. We consider first the error

$$\int_{x_{k-1}}^{x_k} f(x) - dx - A(k)$$

where here $A(k)$ is the area of the trapezoid formed by the points $(x_{k-1}, 0), (x_k, 0), (x_k, f(x_k)), (x_{k-1}, f(x_{k-1}))$.

If you draw a picture, you will see that the above error is exactly the same as the following integral,

$$\int_{x_{k-1}}^{x_k} G(x)dx.$$ 

Where $G(x) = f(x) - (f(x_{k-1}) + (x - x_{k-1}) \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}})$. (This is not at all difficult to see or to conjure up if you draw a picture!) Now, notice that the function $G$ satisfies the conditions of part (a) on the interval $[x_{k-1}, x_k]$. Hence

$$\left| \int_{x_{k-1}}^{x_k} G(x)dx \right| \leq \int_{x_{k-1}}^{x_k} M_2(x - x_{k-1})(x_k - x)dx.$$ 

If you do this integral, you get a quantity that is $O(\delta^3)$ where $\delta = x_k - x_{k-1}$. Now summing over all of the intervals $[x_{k-1}, x_k]$ (there are $\frac{b - a}{\delta}$ such intervals) we get that the total error in the Trapezoid rule is $O(\delta^2)$, as desired.