The Way of Analysis

p. 13:

2.) Write the set $A$ of all finite subsets of $\mathbb{N}$ as the countable union of finite sets

$$A = \bigcup_{i \in \mathbb{N}} A_i,$$

where $A_i$ is the set of all subsets of $\{1, \ldots, i\}$. (Note that $A_i$ has $2^i$ elements.) Since countable unions of countable sets are countable, $A$ is countable.

3.) Since the cartesian product of two countable sets is still countable, the set $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is countable. The map $f : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{Q}$ defined by $f(p, q) = \frac{p}{q}$ is obviously onto. Therefore, $\mathbb{Q}$ is at most countable (it is not finite, so it is countable).

NOTE: You should work out on your own (and if I have time I’ll add it here) a proof of the following claim which gets used in the solution here:

CLAIM: Suppose you have a map $\phi : C \to S$ from a set which is countable onto a set $S$. Then the set $S$ is either finite or it is countable.

4.) Suppose that $B$ is uncountable and $A \subset B$ is countable and define

$$C = B \setminus A = \{x \in B \mid x \notin A\}.$$

Then $B = A \cup C$. Suppose (for contradiction) $C$ were countable, then $B$ would be the union of two countable sets and hence also countable. This is a contradiction. Therefore, $C$ is not countable.

5.) Suppose that a set $A$ has two elements, say $A = \{0, 1\}$. We will show that the countable product of $A$ with itself, call this $B$, is uncountable (this easily implies the claim). There are several ways to prove this, I will do it by showing that $A$ has the same cardinality as $2^\mathbb{N}$. To do this, define a map $f : B \to 2^\mathbb{N}$ by sending $(b_1, b_2, b_3, \ldots)$ (where $b_i$ is 0 or 1) to the subset of $\mathbb{N}$

$$\{j \in \mathbb{N} \mid b_j = 1\}.$$

I will leave it to you to write out the proof that this map is a bijection (“bijection” is the same as “one-to-one and onto”.) Hence $B$ is uncountable.

7.) Suppose that $f : A \to 2^A$. We will show that $f$ is not onto. Define a subset $B$ of $A$ (i.e., $B \in 2^A$) by

$$B = \{a \in A \mid a \notin f(a)\}.$$

By construction an element $a$ is either in $B$ or $f(a)$ but not in both. In particular, $B \neq f(a)$ for any $a$ so that $f$ is not onto. (This is the same as the proof we gave in class.)