PROBLEMS FROM THE STEIN-SHAKARCHI TEXT

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Additional Problems

I) (Thanks to Professor M. Safonov for this problem. This also appeared on our midterm.) Let \( \{f_n\}_{n=1}^{\infty} \) be measurable functions defined on a set \( E \subset \mathbb{R}^d \) with finite measure, \( m(E) < \infty \). Assume that \( f_n(x) \to f(x) \) for almost all \( x \in E \). Suppose also that

\[
\int_E |f_n|^{1+\alpha} \leq C
\]

for some constants \( \alpha > 0 \) and \( C > 0 \).

Prove that,

\[
\int_E |f_n - f| \to 0 \quad \text{as} \quad n \to \infty.
\]

Justify fully your argument. (In particular, if you claim any function is integrable, explain why it is so, citing any limit theorem you use explicitly.)

Hint: Consider truncating functions as we did several times in the class. Indeed consider

\[
F_A(t) \equiv \begin{cases} 
  t & \text{if } |t| < A \\
  A & \text{if } t \geq A \\
  -A & \text{if } t \leq -A.
\end{cases}
\]

Then consider the uniformly bounded functions,

\[
g_{n,A}(x) \equiv F_A(f_n(x)), \quad g_A(x) \equiv F_A(f(x))
\]

II) (Thanks to Professor L. Guth (formerly UToronto) for this problem): Assume \( f \in L^1(\mathbb{R}^2) \) with \( \|f\|_{L^1} = 1 \).

Prove that there exists a point \( x \) in the unit disc \( (|x| \leq 1) \) so that

\[
\int_{\mathbb{R}^2} f(y)|x - y|^{-1} \, dy < 100.
\]