\[ P(x) = 4 + 3x_2 + 4x_1x_2^2 + 2x_1x_2x_3^2 + x_1^2x_2^2 + 2x_1^3x_3^2 + 3x_1^5 \]

\[ a_{0,1,0} = 2, \quad a_{1,1,0} = 1, \quad a_{1,1,1} = -1, \quad a_{2,0,0} = 1, \quad a_{0,2,1} = 5 \]

otherwise \( a_I = 0 \)

\[
\sum_{m=0}^{3} \sum_{I \in \mathcal{I}_n^m} a_I x^I
\]

\[ a_{1,1,0,0} = 3, \quad a_{0,1,1,1} = -1, \quad a_{0,2,1,0} = 2, \quad a_{0,2,0,4} = 1, \quad a_{0,5,0,0} = 1 \]

otherwise \( a_I = 0 \)

\[
\sum_{m=0}^{5} \sum_{I \in \mathcal{I}_n^m} a_I x^I
\]

5. We can see this problem as partitioning the number \( m \) into \( n \) natural numbers. The number of ways to do this is:

\[
|\mathcal{I}_n^m| = \binom{m+n-1}{m}
\]

6a. Since \( a_i = \frac{f^{(i)}(0)}{i!} \) we must show that \( f^{(2k)}(0) = 0 \). We have given that \( f(-x) = -f(x) \). Note that this means \( f(0) = -f(0) = 0 \). We wish to show that \( f^{(2k)}(-x) = -f^{(2k)}(x) \) and we will do this by induction.

\[ k = 0 : \text{This case is true by our assumption.} \]

Inductive Step \( k \rightarrow k+1 : \)

Assume \( f^{(2k)}(-x) = -f^{(2k)}(x) \). We can the differentiate each side of this equation.

\[
-f^{(2k+1)}(-x) = -f^{(2k+1)}(x)
\]

\[
f^{(2(k+1))}(-x) = -f^{(2(k+1))}(x)
\]

So \( f^{(2k)}(-x) = -f^{(2k)}(x) \) is true for all non negative integers \( k \), and \( f^{(2k)}(0) = 0. \)

6b. This proof carries through in precisely the same way as 6a.

9.

\[
f = \frac{x^2(y - x) y}{x^2 + y^2}
\]
As \((x, y) \to (0, 0)\) the derivatives also approach 0. So the first partial derivatives are continuous.

c. 
\[
D_1 D_2 f(0, 0) = \lim_{h \to 0} \frac{D_2 f(h, 0) - D_2 f(0, 0)}{h} = \lim_{h \to 0} \frac{h^4}{h^4} = 1
\]

\[
D_2 D_1 f(0, 0) = \lim_{h \to 0} \frac{D_1 f(0, h) - D_1 f(0, 0)}{h} = 0
\]
d. This does not contradict 3.3.9 as the second partial derivatives are not continuous.

2 3.4

2a. \(f(x, y) = \sin(x + y^2)\) \(P^3_{f,0}(x, y) = x + y^2 + \frac{x^3}{3!}\)

2b. \(f(x, y) = \frac{1}{1 + x^4 + y^2}\) \(P^4_{f,0}(x, y) = 1 - x^2 - y^2 + x^4 + 2x^2y^2 + y^4\)

3. \(F(x, y) = \sqrt{x + y + xy}, \ a = (-2, -3)\)
   
   Set \(g(x, y) = x + y + xy - 1, \ f(u) = \sqrt{1 + u}, \ F(x, y) = f(g(x, y))\)
   
   \(g(-2, -3) = 0 \) \(P^2_{f,0}(u) = 1 + \frac{1}{2} u + \frac{1}{2} \frac{-1}{2} u^2 = 1 + \frac{1}{2} - \frac{u^2}{4}\)
   
   \(P^2_{g,(-2,-3)}(x, y) = g(-2, 3) + D_1 g(-2, -3)(x + 2) + D_2 g(-2, -3)(y + 3) + \frac{1}{2} D_1 D_1 g(-2, -3)(x + 2)^2 + D_1 D_2 g(-2, -3)(x + 3)\)

2) \((y + 3) + \frac{1}{2} D_2 D_2 g(-2, -3)(y + 3)^3 D_1 g = 1 + y \) \(D_2 g = 1 + x \) \(D_1 D_1 g = 0 \) \(D_1 D_2 g = 1 \) \(D_2 D_2 g = 0\)

\[
P^2_{g,(-2,-3)}(P^2_{g,(-2,-3)}(x, y)) = 1 + \frac{-2(x + 2) - (y + 3) + (x + 2)(y + 3)}{8} \frac{(2(x + 2) - (y + 3) + (x + 2)(y + 3))^2}{8}
\]

Throwing out higher order terms we are left with:

\[
1 + \frac{-2(x + 2) - (y + 3)}{4(x + 2)^2 + (y + 3)^2}.
\]

5. 

\[
f(0)
\]

\[
f(h) = f(0) + hf'(0) + \frac{h^2}{2} f''(0) + \frac{h^3}{3!} f'''(0) + ...
\]

\[
f(2h) = f(0) + 2hf'(0) + 2h^2 f''(0) + \frac{4}{3} h^3 f'''(0) + ...\]

\[
\int_0^{2h} f(t) dt = 0 + 2f(0)h + 2h^2 f'(0) + \frac{4}{3} h^3 f''(0) + \frac{16}{24} h^4 f'''(0)
\]
Now we equate each term that has a factor of \(f(0), \ f'(0), \ f''(0)\) and so on.

\[
f(0)(a + b + c) = 2hf(0)
\]

\[
f'(0)h(b + 2c) = 2h^2 f'(0)
\]
\[ f''(0) h^2 \left( \frac{b}{2} + 2c \right) = \frac{4h^3}{3} f''(0) \]

Which leads to the system

\[
\begin{align*}
(a + b + c) &= 2h \frac{b}{2} + 2c = 2h \frac{b}{2} + 2c = \frac{4h^3}{3} \\
(0 + b + c) &= 2h \frac{b}{2} + 2c = \frac{4h^3}{3}
\end{align*}
\]

which leads to \((a, b, c) = \left( \frac{1}{3}, 4, 1 \right)h^3\).