1.8.6 Show the rule for differentiating a dot product. \( f, g : U \to \mathbb{R}^m \)

\[ [D(f \circ g)(a)] \mathbf{v} = [Df(a)] \mathbf{v} \circ g(a) + f(a) \circ [Dg(a)] \mathbf{v} \]

Use rule 5 on each component of \( f \).

b.

4.1.1

If \( C_i \in D_i(\mathbb{R}^2) \), \( Vol C_i = \frac{1}{2^i} \)

4.1.4 Prove proposition 4.1.22. In both cases we are looking at integrating characteristic functions, so \( L_N \geq 0 \). If we show \( U_N \to 0 \) then we are done. If a set \( X \) has volume 0, then \( \int_{X} \chi(x) \, dx = 0 \) which means \( \lim_{N \to \infty} U_n(\chi_X) = U(\chi_X) = 0 \) and \( U_n(\chi_X) = \sum_{C \in D_n(\mathbb{R}^n)} C \cap X \neq \emptyset \) \( Vol_n(C) \) must be arbitrarily small since it approaches 0.

If the second part of the statement is true, then \( U_n \to 0 \) and the volume is 0.

4.1.5

a. \( \sum_{i=0}^{n-1} i = \frac{n(n+1)}{2} \)

b. For the sets \( X_i = [0, 1], [0, 1], (0, 1], (0, 0) \) calculate \( \int x \chi_{X_i} \, dx \)

From Calc I we expect \( \int x \chi_{X_i} \, dx = \frac{1}{2} \). The tricky part here to worry about is that the dyadic cubes are closed on the left and open on the right.

Let’s start with \( X_4 \). Here both ends are open. When we calculate \( U_N \) we find the maximum value on the intervals \( \left( \frac{k}{2^N}, \frac{k+1}{2^N} \right) \). This is non zero for \( k \) from 0 to \( 2^N - 1 \).

\[
U_{4,N} = \frac{1}{2^N} \sum_{i=0}^{2^{N-1}} \frac{i+i+1}{2^N} = \frac{1}{2^N} \left( \frac{2^N(2^N-1)}{2} \right) + \frac{1}{2^N} = \frac{4^N-2^N}{2^{2N}} + \frac{1}{2^N}
\]

\[
\lim_{N \to \infty} U_{4,N} = \frac{1}{2}
\]

\[
L_{4,N} = \frac{1}{2^N} \sum_{i=0}^{2^{N-1}} \frac{i}{2^N} = \frac{1}{2^N} \left( \frac{2^N(2^N-1)}{2} \right) = \frac{4^N-2^N}{2^{2N}}
\]

\[
\lim_{N \to \infty} L_{4,N} = \frac{1}{2}
\]

OK, now let’s look at one where the right endpoint is closed, \( X_2 \). In this case we must also include the next interval, \( \left( \frac{k}{2^N}, \frac{k+1+1}{2^N} \right) \). This only includes one extra value, so

\[
U_{2,N} = U_{4,N} + \frac{1}{2^N}, \lim_{N \to \infty} U_{2,N} = \lim_{N \to \infty} U_{4,N} = \frac{1}{2}
\]

\[
L_{2,N} = L_{4,N} + \frac{1}{2^N}, \lim_{N \to \infty} L_{2,N} = \lim_{N \to \infty} L_{4,N} = \frac{1}{2}
\]

By similar arguments \( U_{3,N} = U_{2,N}, \quad U_{1,N} = U_{4,N} \quad L_{3,N} = L_{2,N} \quad L_{1,N} = L_{4,N} \)

b. Calculate \( \int x \chi_{[a,b]} \, dx \)

Here we will be concerned with the dyadic cubes \( \left[ \frac{k}{2^N}, \frac{k+1}{2^N} \right) \) for \( k \) from 0 to \( K = \lfloor a 2^N \rfloor \).

\[
U_N = \frac{1}{2^N} \sum_{i=0}^{K} \frac{i+i+1}{2^N} = \frac{1}{4^N} \frac{(K(K+1))}{2} + \frac{1}{2^N}
\]

\[
U_N \to \frac{a^2}{2}
\]

\[
L_N \to \frac{a^2}{2}
\]

\[
L_N \to \frac{a^2}{2}
\]

c. All of these integrals can be computed using proposition 4.1.13. For instance

\[
\begin{align*}
\chi_{[0,a]} - \chi_{[0,a]} = \chi_{(a,b)}
\end{align*}
\]

So all of these integrals will be equal to \( \frac{a^2}{2} \).
4.1.6
a. \( \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \)

b. Let \( K_N = [a2^N] \) In all of the cases we may be off in our calculations by a factor of \( \frac{a^2}{2N} \) depending on where \( a \) may fall with respect to the dyadic cubes. We let \( -\frac{a^2}{2N} \leq \epsilon_N \leq \frac{a^2}{2N} \). Note that \( \lim_{N \to \infty} \epsilon_N = 0 \)

\[
U_N = \frac{1}{2N} \sum_{i=0}^{K_N} \left( \frac{i+1}{2N} \right)^2 = \frac{1}{2N} \sum_{i=0}^{K_N+1} i^2 = \frac{1}{2N} \frac{(K_N+1)(K_N+2)(2K_N+3)}{6} + \epsilon_N.
\]

\[
L_N = \frac{1}{2N} \sum_{i=0}^{K_N} \left( \frac{i}{2N} \right)^2 = \frac{1}{2N} \frac{(K_N)(K_N+1)(2K_N+2)}{6} + \epsilon_N.
\]

\[
\lim_{N \to \infty} U_N = \frac{a^3}{3}, \quad \lim_{N \to \infty} L_N = \frac{a^3}{3}
\]

c. All of these integrals can be computed using proposition 4.1.13. For instance

\[
\int x^2 \chi_{[0,b]} - x^2 \chi_{[0,a]} = x^2 \chi_{(a,b)}
\]

So all of these integrals will be equal to \( \frac{b^3-a^3}{3} \).

4.1.9

\[
\nabla f = \langle \cos(x-y), -\cos(x-y) \rangle < |\nabla f| < \sqrt{2}
\]

\[
|f(x_0 + h) - f(x_0)| \leq \epsilon |h| \sqrt{|h|} = |h|(|\epsilon + \sqrt{2}|
\]

On a dyadic cube of \( D_N \), \( |h| < \sqrt{\frac{1.2}{2N} + \frac{1}{2N} \text{osc}_N} \leq \sqrt{\frac{2}{2N}} (\epsilon + \sqrt{2}) \)

\[
U_N - L_N = \sum_{c \in D_N} \text{osc}_C \text{vol} C \leq \frac{\sqrt{2}}{2N} (\epsilon + \sqrt{2})
\]

The right hand side of this inequality goes to 0 as \( N \to \infty \)

4.1.11

Define dilation by

\[
D_a f(x) = f\left(\frac{x}{a}\right).
\]

Show that if \( f \) is integrable then so is \( D_a f \) and

\[
\int D_a f(x) \, dx = 2^n \int f(x) \, dx.
\]

Dilation by \( 2^N \) will map dyadic cubes of \( D_{m+N} \) to a dyadic cube of \( D_m \). If we look at the maximum and minimum values we have

\[
M_C(f) = M_{C'}(D_{2N} f), \quad m_C(f) = m_{C'}(D_{2N} f), C \in D_{m+N}, C' \in D_m
\]

The volume of a cube is multiplied by \( 2^{Nn} \), so

\[
U_m(D_{2N}(f)) = 2^{Nn} U_{m+N}(f)
\]

\[
L_m(D_{2N}(f)) = 2^{Nn} L_{m+N}(f)
\]

\[
\int D_{2N}(f) \, dx = 2^{Nn} \int f(x) \, dx
\]

4.1.14 a. The left and right handed Riemann sums are both equal to zero since all the endpoints are rational numbers.

b. If we use the geometric mean of the endpoints then all of these are irrational, except for \( k = -1, 0, 0 \) so the Riemann sum is \( 2 - \frac{1}{2^{N+1}} \)

4.3.1 a. Let \( f = f^+ - f^- \) \( |f| = f^+ + f^- \), so \( \int |f| \, dx = \int f^+ \, dx + \int f^- \, dx \)

\[
|\int f \, dx| = |\int f^+ \, dx| - |\int f^- \, dx| \geq |\int f^+ \, dx| + |\int f^- \, dx|
\]

\[
= \int |f^+| \, dx + \int |f^-| \, dx = \int |f| \, dx
\]

b. Consider \( f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ -1 & x \in [0,1] \setminus \mathbb{Q} \\ 0 & \text{otherwise} \end{cases} \)

\[
|f(x)| = \chi_{[0,1]} \int |f(x)| \, dx = 1
\]

\[
\int f(x) \, dx \text{ does not exist.}
\]
4.3.2

Show that a subset of a set of volume zero has volume zero.

Let $A \subset B$ with $vol(B) = 0$ which means that $\int \chi_B|dx| = 0$. Since $A \subset B$, then $\chi_A \leq \chi_B$ and we have $\int 0|dx| \leq \int \chi_A|dx| \leq \int \chi_B|dx| = 0$, $\int \chi_A|dx| = 0$

4.3.5 The region is bounded by the equations $y = x^2, y = 1$ and the function is continuous on this region, so by corollary 4.3.11 it is integrable.