MATH 3283W
Spring 2016
Exam 1
Thursday 11 February 2016
Time Limit: 50 minutes

Name (Print): Solution & grading guide
Student ID:
Section Number:
Teaching Assistant:
Signature:

This exam contains 5 numbered problems on three sheets of paper. Check to see if any pages are missing. Point values are in parentheses. No books, notes, or electronic devices are allowed.

As on the writing quizzes, your work will be graded on the quality of your writing as well as on the validity of the mathematics. In particular, included in the 20 points for problem #5 is a 5-point writing score.

Except on 1(a) and 1(b), where they are explicitly allowed, do not use symbols for logical connectives and quantifiers. That is, do not use the symbols $\Rightarrow$, $\Leftrightarrow$, $\wedge$, $\vee$, $\sim$, $\forall$, $\exists$, and $\ni$.

1	20 pts	
2	20 pts	
3	20 pts	
4	20 pts	
5	20 pts	
TOTAL	100 pts	

1. (20 points) (5 points each) On 1(a) and 1(b) only, you may use symbols for logical connectives.

(a) Write a truth table for the conditional statement $p \Rightarrow q$. You may use the symbols T/F or 1/0 for true/false, respectively.

layout 1

p	q	$p \Rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

1 pt per line

(b) Write one choice of truth values for p, q, and r (again, using T/F or 1/0), not an entire truth table, to show that the statements

$$p \wedge (q \vee r) \qquad \text{and} \qquad (p \wedge q) \vee r$$

are not logically equivalent.

1 each (Choose $p = 0$, $q = 1$ ($q = 0$ also works) and $r = 1$.

1 each (Then $p \wedge (q \vee r)$ is false ($= 0$) and $(p \wedge q) \vee r$ is true ($= 1$).

(c) Give a counterexample to this false statement: For all real numbers x and y, if x and y are irrational, then $x + y$ is irrational.

2 (Choose $x = \sqrt{2}$ and $y = -\sqrt{2}$.

1 (Then x and y are irrational,

2 (but $x + y = 0$ is rational.

(d) Define $A_n = (3, 4 + \frac{1}{n})$, an open interval in $\mathbb{R}$, for each natural number n. Without writing a proof, find

$$\bigcap_{n=1}^{\infty} A_n. = (3, 4]$$

1 2

2. (20 points) Consider the following statement about real numbers x, y, and z:

For all x, there exists y such that for all z, if $y < x$ then $z < y$.

(a) (10 points) Write the negation of the statement above.

1 There exists x such that

1 for all y,

1 there exists z such that

$\underbrace{y < x}_{2}$ $\underbrace{\text{and}}_{2}$ $\underbrace{z \geq y}_{3}$.

(b) (10 points) Determine whether the original statement is true or false. Write "true" or "false", and then justify your answer by proving the original statement or the negation that you wrote in (a).

TRUE. Proof: Let x be given. 1

[Grade structure independent of T/F answer.]

Choose $y = x + 1$. 1

Let z be given. 1

7 We prove the contrapositive: if $z \geq y$, then $y \geq x$.

[Suppose $z \geq y$.]

Then $y = x + 1 \geq x$.

Alternately observe that $y < x$ is false, and hence the conditional is true.

3. (20 points) (a) (6 points) Complete the following definition, which should be in the form of a universal statement: We say that a relation $\sim$ on a set A is *transitive* if ...

2 (for all $x, y, z \in A$,

2 (if $x \sim y$ and $y \sim z$

2 (then $x \sim z$.

(b) (8 points) Write the negation of your definition in (a). Your answer should be in the form of an existential statement.

2 (There exist $x, y, z \in A$ such that

2 ($x \sim y$ and $y \sim z$

2 (and

2 ($x \not\sim z$.

(c) (6 points) Let $\sim$ be the relation on $\mathbb{R}^2$ defined as follows: we define $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 = x_2$. This relation is in fact an equivalence relation (which you do not need to prove). Write one complete sentence that describes *geometrically* the equivalence classes of $\sim$.

Each vertical line in $\mathbb{R}^2$ is an equivalence class.

either okay

Each equivalence class is a vertical line in $\mathbb{R}^2$.

sentence 1 3 2

4. (20 points) Consider the following two statements. One is true, and one is false.

A. For all subsets S and T of a universal set U, we have $U-(S-T) \subseteq (U-S) \cup T$.

B. For all subsets S and T of a universal set U, we have $U-(S-T) \subseteq (U-S) \cap T$.

(a) (12 points) Identify the true statement. That is, write "A is true" or "B is true". Then give a proof of the true statement.

Notes: We use the notation $S-T$ for the *complement* of T in S, whereas the textbook uses the notation $S\backslash T$. Prove directly from the definitions of the set operations; do not use any previously proven facts about sets. You may use logical equivalences without citing or proving them.

A is true. Proof: (2) Suppose $x \in U-(S-T)$.

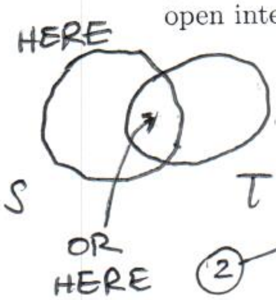
(2) That is, $x \in U$ and $x \notin S-T$.

(4) Since $x \notin S-T$, we have $x \notin S$ or $x \in T$.

(Case 1) if $x \notin S$, then $x \in U-S$ and hence $x \in (U-S) \cup T$. (2)

(Case 2) Otherwise, $x \in T$ and hence $x \in (U-S) \cup T$. (2)

(b) (8 points) Give a counterexample to the false statement, where $U = \mathbb{R}$, and S and T are open intervals in $\mathbb{R}$.



Choose $S = \underbrace{(0,2)}_{(2)}$ and $T = \underbrace{(1,3)}_{(2)}$.

(2) $-0.5 \in U-(S-T)$ but

(2) $-0.5 \notin (U-S) \cap T$.

(Alternately, use 1.5)

(Any choice of S, T works, provided $x \notin S \cup T$.)

5. (20 points) (Five of the 20 points will be for a writing score, as described on the course website.) Prove that, for all integers p and q, if pq is an even integer, then p is an even integer or q is an even integer.

3 Proof: We prove the contrapositive:

3 if p is odd and q is odd, then pq is odd.

Suppose that p is odd and q is odd.

3 That is, $p = 2k + 1$, for some integer k,

3 and $q = 2\ell + 1$, for some integer ℓ.

different label

Then
$$
\begin{aligned}
pq &= (2k+1)(2\ell+1) \\
&= 4k\ell + 2k + 2\ell + 1 \\
&= 2(2k\ell + k + \ell) + 1.
\end{aligned}
$$
3

Thus pq is odd. $\lrcorner$