

MATH 3283W
Fall 2017
Exam 1
Thursday, September 28, 2017
Time Limit: 50 minutes

Name (Print): _____
Student ID: _____
Section Number: _____
Teaching Assistant: _____
Signature: _____

[Handwritten Signature] Solomon Ginde

This exam contains 5 numbered problems on 5 sheets of paper. In addition to those 5 pages there is a cover sheet (this page) and a blank sheet at the back - for a total of 7 pages. Please check to see if any pages are missing before you begin your work. No books, notes, or electronic devices are allowed.

As on the writing quizzes, your work will be graded on the quality of your writing as well as on the validity of the mathematics. In particular, included in the 20 points for problem #5 is a 5-point writing score.

Except on 1(a) and 1(b), where they are explicitly allowed, do not use symbols for logical connectives and quantifiers. That is, do not use the symbols \Rightarrow , \Leftrightarrow , \wedge , \vee , \sim , \forall , \exists , and \ni .

1	20 pts	
2	20 pts	
3	20 pts	
4	20 pts	
5	20 pts	
TOTAL	100 pts	

1. (20 points) (5 points each) On 1(a) and 1(b) only, you may use symbols for logical connectives.

(a) Write a truth table for the following statement,

$$(\sim q) \Rightarrow (\sim p).$$

You may use the symbols T/F or 1/0 for true/false, respectively.

p	q	$\sim q$	$\sim p$	$(\sim q) \Rightarrow (\sim p)$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

(b) Indicate whether the following statement is true or false (write "TRUE" or "FALSE" very clearly) and then give a brief justification for your answer:

If $4 + 4 = 8$ then 9 is prime.

False.

The antecedent (" $4+4=8$ ") is true, but the consequent ("9 is prime") is false. A conditional statement with true antecedent and false consequent is false. (i.e., if p is true and q is false, then $p \rightarrow q$ is false)

(c) Define $A_n = [n, n+1] \subset \mathbb{R}$, a closed interval in \mathbb{R} , for each natural number n . Without writing a proof, find $\bigcap_{n=1}^{\infty} A_n$.

$$\bigcap_{n=1}^{\infty} A_n = \emptyset \text{ since } \{A_n\}_{n=1}^{\infty} \text{ contains disjoint intervals}$$

(d) Define $B_n = [3-n, 7]$, a closed interval in \mathbb{R} for each natural number n . Without writing a proof find $\bigcap_{n=1}^{\infty} B_n$.

$$\bigcap_{n=1}^{\infty} B_n = [2, 7] \text{ since } B_1 = [2, 7] \text{ and } B_1 \subset B_n \text{ for all } n > 1$$

(e) Same definitions as in part (d) above: Without writing a proof find $\bigcup_{n=1}^{\infty} B_n$.

$$\bigcup_{n=1}^{\infty} B_n = (-\infty, 7] \text{ since as } n \rightarrow \infty, 3-n \rightarrow -\infty.$$

2. (20 points) Consider the following statement about real numbers x , y , and z :

For all x , there exists y such that for all z , if $z > y$ then $z > x + y$.

(a) (6 points) Write the negation of the statement above.

There exists x such that for all y ,
there exists z with
 $z > y$, and $z \leq x + y$.

(b) (14 points) Determine whether the original statement is true or false. Write "true" or "false", and then justify your answer by proving the original statement or the negation that you wrote in (a).

False.

Proof of the negation. Let $x=1$, and y is an arbitrary real number. For each y , set $z = y+1$. Then

$1 > 0$ implies $y+1 > y$, or in other words, $z > y$.

On the other hand,

$$z = y+1 = y+x \leq x+y.$$

Hence, we've shown that there exists x (namely, $x=1$), such that for each y , there exists z satisfying $z > y$, and $z \leq x+y$. \square

Caution! x cannot depend on y nor z !

3. (20 points) Define a relation on \mathbb{R} by,

$$x \sim y \Leftrightarrow (\exists p \in \mathbb{R} \text{ so that } x - y = \sin(p)).$$

(a) (10 points) Is this an equivalence relation on \mathbb{R} ? If so prove it, labelling your argument carefully. If it is not, explain why not. (Your answer must begin with the word "YES" or the word "NO", depending on whether you claim this is an equivalence relation or not.)

No, This is not an equivalence relation. It fails the transitive property!

for example but $x=1, y=2, z=3$.

Then $x \sim y$ because $x - y = -1 = \sin(-\frac{\pi}{2})$.
 $y \sim z$ " $y - z = -1 = \sin(-\frac{\pi}{2})$

But x and z are not related since $x - z = -2$ and

(b) (10 points) Define a relation on $A = (\mathbb{Z} \setminus \{0\}) \times \mathbb{Z}$ by

$$(p, q) \sim (p', q') \Leftrightarrow \frac{q}{p} = \frac{q'}{p'}$$

there is no number p for which $\sin(p) = -2$

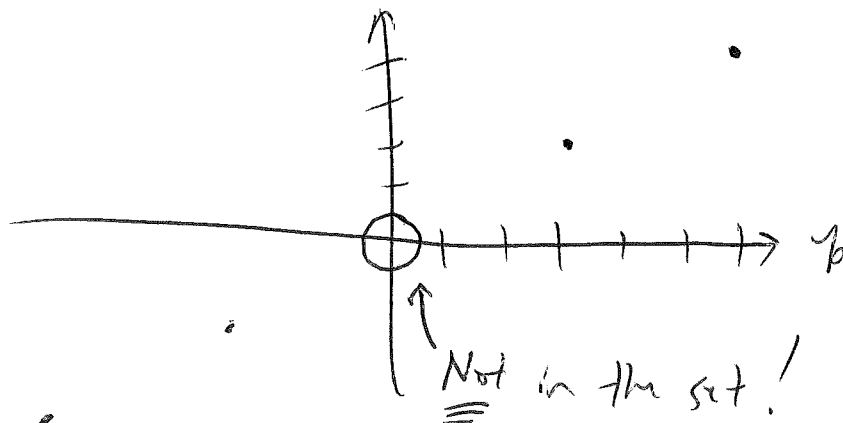
This is an equivalence relation on the set A (you need not prove this fact). Describe and sketch the equivalence class

$$A_{(3,2)}$$

That is, sketch the equivalence class of the point $(3, 2) \in A$.

$$A_{(3,2)} = \left\{ (p, q) \in A \mid \frac{q}{p} = \frac{2}{3} \right\}$$

$$= \left\{ (p, q) \in \mathbb{Z} \times \mathbb{Z} \mid p \neq 0 \text{ and } q = \frac{2}{3}p \right\}$$



← THE set is a subset of A line through origin!

4. (20 points) Consider the following two statements. One is true, and one is false.

A. For all nonempty sets A and B , we have $A \setminus (A \setminus B) = B \setminus (B \setminus A)$.

B. For all nonempty sets A and B , we have $A \setminus (B \setminus A) = B \setminus (A \setminus B)$.

(a) (14 points) Identify the true statement. That is, write "A is true" or "B is true". Then give a proof of the true statement.

Notes: Prove directly from the definitions of the set operations; do not use any previously proven facts about sets. You may use logical equivalences without citing or proving them.

A is true. To show $A \setminus (A \setminus B) = B \setminus (B \setminus A)$, we first pick $x \in A \setminus (A \setminus B)$. Then $x \in A$ and $x \notin A \setminus B$. But $x \notin A \setminus B$ means $x \in B$. Since we know already $x \in A$. We have shown that $x \in A \cap B$. Hence $x \in B$ and $x \notin B \setminus A$ which implies $x \in B \setminus (B \setminus A)$.

WE HAVE SHOWN $A \setminus (A \setminus B) \subseteq B \setminus (B \setminus A)$.

If we interchange the roles played by "A" and "B" in the above argument (writing exactly the same thing but switching "A" and "B"), we conclude also $B \setminus (B \setminus A) \subseteq A \setminus (A \setminus B)$.

(b) (6 points) Give a counterexample to the false statement, in the case where A and B are nonempty open intervals in \mathbb{R} . (Make sure to demonstrate why your example really is a counterexample.)

There are lots of possible counterexamples,

for example:

$$A = (0, 1) \subseteq \mathbb{R}$$

$$B = (2, 3) \subseteq \mathbb{R}$$

THEN $A \setminus (B \setminus A) = A = (0, 1)$ and $(0, 1) \neq (2, 3)$
 $B \setminus (A \setminus B) = B = (2, 3)$

Together
 these
 conclusions
 give the
 equality.
 We wanted
 to show!

5. (20 points) (Five of the 20 points will be for a writing score, as described in your section meetings and on the course website.)

True or false: If x is an irrational number and y is a rational number, then $x + y$ is an irrational number.

Write "TRUE" or "FALSE" and justify your answer. If you wrote "TRUE" you should prove the statement. (If your proof uses some result about rational or irrational numbers beyond the definitions of these sets, you must reprove that result in your solution.) If you wrote "FALSE" you should give a counterexample.

Claim: If x is irrational and y is rational, then $x+y$ is irrational.

This statement is TRUE.

Proof: Let x be irrational, and let y be rational, which is to say that

$y = \frac{a}{b}$ for $a, b \in \mathbb{Z}$, $b \neq 0$. Assume, to the contrary, that $x+y$ is rational.

Then $x+y = \frac{c}{d}$ for $c, d \in \mathbb{Z}$, $d \neq 0$, by definition of a rational number.

$$\text{Thus } x = x+y - y = \frac{c}{d} - \frac{a}{b} = \frac{cb - ad}{bd}.$$

Since $a, b, c,$ and d are integers, $cb - ad$ is also an integer, as is bd .

Also, since $b \neq 0$ and $d \neq 0$, $bd \neq 0$ must also hold.

Thus $x = \frac{cb - ad}{bd}$ means that x is rational. This is a contradiction,

which completes the proof that $x+y$ is irrational.