Solution Guide

MATH 3283W
Spring 2018
Exam 1
Thursday, February 8, 2018
Time Limit: 50 minutes

Name (Print): ______________________
Student ID: ______________________
Section Number: ______
Teaching Assistant: ______________________
Signature: ______________________

This exam contains 6 numbered problems on 6 sheets of paper. In addition to those 6 pages there is a cover sheet (this page) and a blank sheet at the back - for a total of 8 pages. Please check to see if any pages are missing before you begin your work. No books, notes, or electronic devices are allowed.

As on the writing quizzes, your work will be graded on the quality of your writing as well as on the validity of the mathematics. In particular, included in the 20 points for problem #6 is a 5-point writing score.

Except on 1(a) and 1(b), where they are explicitly allowed, do not use symbols for logical connectives and quantifiers. That is, do not use the symbols $\Rightarrow$, $\Leftrightarrow$, $\wedge$, $\vee$, $\sim$, $\forall$, $\exists$, and $\ni$.

1	20 pts	
2	15 pts	
3	15 pts	
4	15 pts	
5	15 pts	
6	20 pts	
TOTAL	100 pts	

1. (20 points) On 1(a) and 1(b) only, you may use symbols for logical connectives.

(a) (4pts.) Write a truth table for the following statement,

$$\sim (p \Rightarrow q).$$

You may use the symbols T/F or 1/0 for true/false, respectively.

p	q	$p \Rightarrow q$	$\sim(p \Rightarrow q)$
T	T	T	F
F	F	T	F
F	T	T	F
T	F	F	T

(b) (6 pts). Construct a truth table to prove the following:

$$\sim (p \vee q) \iff (\sim p) \wedge (\sim q)$$

p	q	$p \vee q$	$(\sim p) \wedge (\sim q)$	$\sim(p \vee q)$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

(c) (2 pts). Write the negation of the following statement: "Every ham sandwich is delicious."

There exists a ham sandwhich that is not delicious.

(d) (4 pts) Write the negation of the following statement: If $y \geq 9$, then $(g(x) < 6$ or $g(x) > 13)$.

There exists a $y \geq 9$ with $g(x) \geq 6$ and $g(x) \leq 13$.

(e).(4 pts). Consider this statement: If 3 is odd or $4 > 6$, then $9 \leq 5$.

Is that statement True or False? Write clearly 'True' or 'False' then give some brief explanation for your answer.

False. The conclusion ("$9 \leq 5$") is false while the hypothesis "3 is odd or $4>6$" is true.

2. (15 points) (a). (4 pts). Write the contrapositive of the following statement:

If f is continuous and C is connected, then $f(C)$ is connected.

If $f(C)$ is not connected, then either f is not continuous or C is not connected.

(b). (11 pts). If the following statement is true, prove it. If it is false give a counterexample.

For all x there exists a y so that if $x > 0$ then we have the following two things are true:

$$y \leq 0 \text{ and } y^2 = x.$$

The statement is true.

Proof. Fix $x \in \mathbb{R}$. If $x \leq 0$, take $y = 0$. Then the statement

"if $x > 0$, then ($y \leq 0$ and $y^2 = x$)"

is true since the hypothesis "$x > 0$" is falsified. Otherwise, $x > 0$. In this case, there exists a real number z such that $z^2 = x$ and $z > 0$, which we denote as $\sqrt{x}$. Take $y = -\sqrt{x}$. Since $\sqrt{x} > 0$, then $-\sqrt{x} < 0$, so that $y \leq 0$, and

$$y^2 = (-\sqrt{x})(-\sqrt{x}) = (\sqrt{x})^2 = x,$$

as desired.

3. (15 points) Consider the following statement about real numbers x, y, and z:

For all x, there exists y so that for all z, $x + y = z$.

(a) (4 points) Write the negation of the statement above.

There exists an x such that for all y there exists a z such that $x+y \neq z$.

(b) (11 points) Determine whether the original statement is true or false. Write "true" or "false", and then justify your answer by proving the original statement or the negation that you wrote in (a).

The orignal statement is FALSE!

Proof of (a): Let $x=0$. Given y let $z=y-1$. Then

$$x+y=y \neq y-1=z. \blacksquare$$

4. (15 points) Prove the following statement:

$$\text{If } \frac{x}{x-2} \leq 3 \text{ then } x < 2 \text{ or } x \geq 3.$$

Suppose $\frac{x}{x-2} \leq 3$. Let's split into two cases.

Case 1: Suppose also $x < 2$. If this is the case, we have nothing more to show since we have the conclusion $x < 2$ or $x \geq 3$.

Case 2: Suppose also that $x \geq 2$.

Notice that $x = 2$ is impossible (in that case the left side of the inequality in the assumption is not defined). Hence we have $x > 2$, so $x - 2 > 0$.

This gives us that $x \leq 3(x-2)$

Which implies $x \leq 3x - 6$.

We conclude that $6 \leq 2x$, ~~and~~

so $3 \leq x$.

5. (15 points) Consider the following statement: For every positive integer n, we have that $n^2 + 3n + 8$ is even.

   State whether this is True or False. If you say it is True, prove it. If you say it is False, give a counterexample.

<u>True.</u>

We have $n^2+3n+8 = n(n+3) + 8$.

If n is even, then $n \cdot (n+3)$ is even since it is the product of an even and an odd integer.

Hence $n \cdot (n+3) + 8$ is the sum of two even integers and so it is even too!

If n is odd, then $n+3$ is even so $n(n+3)$ is also even. As before, we conclude $n(n+3)+8$ is even.

6. (20 points) (Five of the 20 points will be for a writing score, as described in your section meetings and on the course website.)
True or false: If p is an integer and p^2 is odd, then p must also be odd.
Write "TRUE" or "FALSE" and justify your answer. If you wrote "TRUE" you should prove the statement. If you wrote "FALSE" you should give a counterexample.

You can prove this using contradiction or contrapositive - the proofs are nearly identical.

Proof by contradiction: Assume p is an integer and p^2 is odd. For the sake of contradiction, assume p is an even integer. Since p is even, there exists an integer k such that $p = 2k$. Then

$$p^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

Since $2k^2$ is an integer, p^2 is even, which is a contradiction. Therefore p must be odd.