

MATH 3283W

Spring 2017

Exam 2

Thursday 9 March 2017

Time Limit: 50 minutes

Name (Print): _____

Student ID: _____

Section Number: _____

Teaching Assistant: _____

Signature: _____

This exam contains 5 numbered problems on four sheets of paper. The last sheet is blank. Check to see if any pages are missing. Point values are in parentheses. No books, notes, or electronic devices are allowed.

As on the writing quizzes, your work will be graded on the quality of your writing as well as on the validity of the mathematics. In particular, a five-point writing score (W) will be determined from your work on problems 3, 4b, and 5b.

Do not use symbols for logical connectives and quantifiers. That is, do not use the symbols \Rightarrow , \Leftrightarrow , \wedge , \vee , \sim , \forall , \exists , and \ni .

Axioms for Ordered Fields

A1. For all $x, y \in \mathbb{F}$, $x + y \in \mathbb{R}$, and if $x = w$ and $y = z$ then $x + y = w + z$.

A2. For all $x, y \in \mathbb{F}$, $x + y = y + x$.

A3. For all $x, y, z \in \mathbb{F}$, $x + (y + z) = (x + y) + z$.

A4. There is a unique element $0 \in \mathbb{F}$ such that $x + 0 = x$ for all $x \in \mathbb{F}$.

A5. For each $x \in \mathbb{F}$, there exists a unique element y such that $x + y = 0$. (Often, we write $y = -x$.)

M1. For all $x, y \in \mathbb{F}$, $x \cdot y \in \mathbb{R}$, and if $x = w$ and $y = z$ then $x \cdot y = w \cdot z$.

M2. For all $x, y \in \mathbb{F}$, $x \cdot y = y \cdot x$.

M3. For all $x, y, z \in \mathbb{F}$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

M4. There is a unique element $1 \neq 0$ in \mathbb{F} such that $x \cdot 1 = x$ for all $x \in \mathbb{F}$.

M5. For each $x \neq 0$ in \mathbb{F} , there exists a unique element y such that $x \cdot y = 1$. (Often, we write $y = 1/x$.)

DL. For all $x, y, z \in \mathbb{F}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

O1. There is a relation $<$ such that, for all $x, y \in \mathbb{F}$, exactly one relation holds: $x = y$, $x < y$ or $y < x$.

O2. For all $x, y, z \in \mathbb{F}$, if $x < y$ and $y < z$ then $x < z$.

O3. For all $x, y, z \in \mathbb{F}$, if $x < y$ then $x + z < y + z$.

O4. For all $x, y, z \in \mathbb{F}$, if $x < y$ and $0 < z$ then $x \cdot z < y \cdot z$.

1	20 pts	
2	15 pts	
3	20 pts	
4	20 pts	
5	20 pts	
W	5 pts	
TOTAL	100 pts	

1. (20 points) (8 points) (a) Write a formula for a bijection $f : \mathbb{N} \rightarrow \mathbb{Z}$. You do not need to show that the function is a bijection.

- (b) (6 points) In section we discussed that, given any set S , there is no surjective function

$$f : S \rightarrow \mathcal{P}(S).$$

In one possible proof by contradiction, we suppose that there is one, we define

$$V = \{x \in S : x \notin f(x)\},$$

and then we show that a contradiction arises.

For example, suppose that $S = \{1, 2, 3, 4, 5\}$ and suppose that $f(1) = \{5, 3, 1\}$, $f(2) = S$, $f(3) = \{2, 4\}$, $f(4) = \{3, 5\}$, and $f(5) = \emptyset$.

In this particular case, **write the set V** . That is, list its elements explicitly.

- (c) (6 points) In the field \mathbb{F} of rational functions with the ordering as defined in class, determine which of the following two rational functions is larger, and explain.

$$\frac{x}{x-2}$$

$$\frac{2x}{x+2}$$

2. (20 points) Suppose that A is a subset of \mathbb{R} that is nonempty and bounded above.
- (a) (8 points) Complete the definition by writing two statements that involve inequalities: we say that $x = \sup A$ if ...

- (b) (12 points) By analogy with our definition of the set $A + B$, let us define the set

$$A^2 = \{a^2 : a \in A\}.$$

Consider the following statement: given a subset A of \mathbb{R} that is nonempty and bounded above, we have

$$\sup(A^2) = (\sup A)^2.$$

If the statement is true, prove it. If it is false, write a counterexample.

3. (20 points) (a) (10 points) Use the axioms of an ordered field \mathbb{F} given on the exam cover page to show that for any $x, y \in \mathbb{F}$, $x < y$ if and only if $-y < -x$.

- (b) (10 points) Prove that any finite intersection of open sets is open.

4. (20 points) Suppose that A and B are sets and that $f : A \rightarrow B$ is a function.
- (a) (5 points) Complete the definition: we say that f is *surjective* if ...

- (b) (15 points) Now suppose that D_1 and D_2 are subsets of B . Prove that

$$f^{-1}(D_1 \cap D_2) \subseteq f^{-1}(D_1) \cap f^{-1}(D_2).$$

5. (20 points) Consider the following two claims:

For all natural numbers n ,

$$\sum_{i=1}^n i^2 = \frac{(n+1)(7n-4)}{6}.$$

For all natural numbers n ,

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

One of the statements is true, and one is false.

(a) (5 points) Give a counterexample to the false statement. That is, find a natural number n for which the equation is not true.

(b) (15 points) Prove the true statement. (Note: the inductive step contains some involved polynomial algebra. Rather than attempting to factor a polynomial, you might try to expand the desired result on the blank page and “meet in the middle”.)

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