MATH 3283W	**Name (Print):** ____________
Fall 2017	**Student ID:** ____________
Exam 2	**Section Number:** ______
Thursday 26 October 2017	**Teaching Assistant:** ____________
Time Limit: 50 minutes	**Signature:** ____________

This exam contains 5 numbered problems on seven sheets of paper. The last sheet is blank. Check to see if any pages are missing. Point values are in parentheses. No books, notes, or electronic devices are allowed.

As on the writing quizzes, your work will be graded on the quality of your writing as well as on the validity of the mathematics. In particular, included in the 20 points for problem 4 is a five-point writing score.

Do not use symbols for logical connectives and quantifiers. That is, do not use the symbols $\Rightarrow$, $\Leftrightarrow$, $\wedge$, $\vee$, $\sim$, $\forall$, $\exists$, and $\ni$.

Axioms for Ordered Fields

A1. For all $x, y \in \mathbb{F}$, $x + y \in \mathbb{F}$, and if $x = w$ and $y = z$ then $x + y = w + z$.
A2. For all $x, y \in \mathbb{F}$, $x + y = y + x$.
A3. For all $x, y, z \in \mathbb{F}$, $x + (y + z) = (x + y) + z$.
A4. There is a unique element $0 \in \mathbb{F}$ such that $x + 0 = x$ for all $x \in \mathbb{F}$.
A5. For each $x \in \mathbb{F}$, there exists a unique element y such that $x + y = 0$. (Often, we write $y = -x$.)

M1. For all $x, y \in \mathbb{F}$, $x \cdot y \in \mathbb{F}$, and if $x = w$ and $y = z$ then $x \cdot y = w \cdot z$.
M2. For all $x, y \in \mathbb{F}$, $x \cdot y = y \cdot x$.
M3. For all $x, y, z \in \mathbb{F}$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
M4. There is a unique element $1 \neq 0$ in $\mathbb{F}$ such that $x \cdot 1 = x$ for all $x \in \mathbb{F}$.
M5. For each $x \neq 0$ in $\mathbb{F}$, there exists a unique element y such that $x \cdot y = 1$. (Often, we write $y = \frac{1}{x}$.)

DL. For all $x, y, z \in \mathbb{F}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

O1. There is a relation $<$ such that, for all $x, y \in \mathbb{F}$, exactly one relation holds: $x = y$, $x < y$ or $y < x$.
O2. For all $x, y, z \in \mathbb{F}$, if $x < y$ and $y < z$ then $x < z$.
O3. For all $x, y, z \in \mathbb{F}$, if $x < y$ then $x + z < y + z$.
O4. For all $x, y, z \in \mathbb{F}$, if $x < y$ and $0 < z$ then $x \cdot z < y \cdot z$.

1	20 pts	
2	20 pts	
3	10 pts	
4	20 pts	
5	10 pts	
TOTAL	80 pts	

1. (20 points) Are the following three statements True or False? If you believe a statement is False, you must start your answer with the word “False” and then give a counterexample (and explain fully why it is a counterexample). If you believe a statement is True, start your answer with the word “True” and then give a proof. (In what follows, $\mathrm{cl}(S)$ denotes the closure of a set $S \subseteq \mathbb{R}$.)

   (a) (7 points)True or False: For any subsets A, B of $\mathbb{R}$,

   $$\mathrm{cl}(A \cap B) \;=\; \mathrm{cl}(A) \cap \mathrm{cl}(B).$$

   (b) (7 points) Suppose A, B, C are three sets. Suppose $f : A \to B$ is surjective. Suppose also that $g : B \to C$ is surjective. True or False: we can conclude that the composition $g \circ f : A \to C$ is always surjective.

   (c) (6 points) True or False: The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x + 3$ is injective.

2. (20 points) Are the following two statements True or False? If you believe a statement is False, you must start your answer with the word "False" and then give a counterexample (and explain fully why it is a counterexample). If you believe a statement is True, start your answer with the word "True" and then give a proof.

(a) (7 points) If S is any set, write $\mathbb{P}(S)$ to denote the power set of S - that is , the set of all subsets of S. True or False: For any sets A, B, we have,

$$\mathbb{P}(A \setminus B) \;=\; \mathbb{P}(A) \setminus \mathbb{P}(B).$$

(b) (13 points) Recall that if we say a set is 'countable', we mean that the set is either finite, or it is denumerable (in 1-1 correspondance with $\mathbb{N}$). Let $A_1, A_2, \ldots, A_n, \ldots$ be a countable collection of sets and assume each A_i is a finite set. True or False: we can conclude that the countable Cartesian product of these sets is also countable, where this product is defined by,

$$A_1 \times A_2 \times \ldots \times A_n \times \ldots \;=\; \big\{(a_1, a_2, a_3, \ldots) \mid a_i \in A_i \text{ for } i \in \mathbb{N}\big\}.$$

3. (10 points) Prove using induction that $9^n - 4^n$ is a multiple of 5 for all $n \in \mathbb{N}$.

4. (20 points) For this problem, you are allowed to use the axioms of ordered fields that appear on the cover of this exam **and you are also allowed to assume the following two results about real numbers without proving these two results:**

   (I). If $p \in \mathbb{R}$ and $p < 0$ then $p = -q$ for some $q > 0$.

   (II). For all $p \in \mathbb{R}, -p = (-1) \cdot p$.

   (15 points) Prove the following statement about real numbers using only the assumptions described above: If $x \neq 0$ then $x^2 > 0$. (Here $x^2 = x \cdot x$.)

   (Your score on this problem will include a writing score of between 0 and 5 points.)

5. (10 points) Is the following statement True or False? If you believe it is False, you must start your answer with the word "False" and then give a counterexample (and explain fully why it is a counterexample). If you believe the statement is True, start your answer with the word "True" and then give a proof.

   If $A \subseteq \mathbb{R}$ is nonempty, open and bounded, then we can always conclude that $\sup(A) \notin A$. (The symbol $\notin$ here denotes 'is not an element of'.)

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