Key

MATH 3283W	**Name (Print):** Key
Fall 2017	**Student ID:**
Exam 2	**Section Number:**
Thursday 26 October 2017	**Teaching Assistant:**
Time Limit: 50 minutes	**Signature:**

This exam contains 5 numbered problems on seven sheets of paper. The last sheet is blank. Check to see if any pages are missing. Point values are in parentheses. No books, notes, or electronic devices are allowed.

As on the writing quizzes, your work will be graded on the quality of your writing as well as on the validity of the mathematics. In particular, included in the 20 points for problem 4 is a five-point writing score.

Do not use symbols for logical connectives and quantifiers. That is, do not use the symbols $\Rightarrow$, $\Leftrightarrow$, $\wedge$, $\vee$, $\sim$, $\forall$, $\exists$, and $\ni$.

Axioms for Ordered Fields

A1. For all $x, y \in \mathbb{F}$, $x + y \in \mathbb{F}$, and if $x = w$ and $y = z$ then $x + y = w + z$.
A2. For all $x, y \in \mathbb{F}$, $x + y = y + x$.
A3. For all $x, y, z \in \mathbb{F}$, $x + (y + z) = (x + y) + z$.
A4. There is a unique element $0 \in \mathbb{F}$ such that $x + 0 = x$ for all $x \in \mathbb{F}$.
A5. For each $x \in \mathbb{F}$, there exists a unique element y such that $x + y = 0$. (Often, we write $y = -x$.)

M1. For all $x, y \in \mathbb{F}$, $x \cdot y \in \mathbb{F}$, and if $x = w$ and $y = z$ then $x \cdot y = w \cdot z$.
M2. For all $x, y \in \mathbb{F}$, $x \cdot y = y \cdot x$.
M3. For all $x, y, z \in \mathbb{F}$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
M4. There is a unique element $1 \neq 0$ in $\mathbb{F}$ such that $x \cdot 1 = x$ for all $x \in \mathbb{F}$.
M5. For each $x \neq 0$ in $\mathbb{F}$, there exists a unique element y such that $x \cdot y = 1$. (Often, we write $y = \frac{1}{x}$.)

DL. For all $x, y, z \in \mathbb{F}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

O1. There is a relation $<$ such that, for all $x, y \in \mathbb{F}$, exactly one relation holds: $x = y$, $x < y$ or $y < x$.
O2. For all $x, y, z \in \mathbb{F}$, if $x < y$ and $y < z$ then $x < z$.
O3. For all $x, y, z \in \mathbb{F}$, if $x < y$ then $x + z < y + z$.
O4. For all $x, y, z \in \mathbb{F}$, if $x < y$ and $0 < z$ then $x \cdot z < y \cdot z$.

1	20 pts	
2	20 pts	
3	10 pts	
4	20 pts	
5	10 pts	
TOTAL	80 pts	

1. (20 points) Are the following three statements True or False? If you believe a statement is False, you must start your answer with the word "False" and then give a counterexample (and explain fully why it is a counterexample). If you believe a statement is True, start your answer with the word "True" and then give a proof. (In what follows, $\mathrm{cl}(S)$ denotes the closure of a set $S \subseteq \mathbb{R}$.)

(a) (7 points)True or False: For any subsets A, B of $\mathbb{R}$,

$$\mathrm{cl}(A \cap B) = \mathrm{cl}(A) \cap \mathrm{cl}(B).$$

FALSE: Take e.g. $A = (0,1)$, $B = (1,2)$.

Then $cl(A) = [0,1]$. $cl(B) = [1,2]$

THEN $A \cap B = \phi$ so $cl(A \cap B) = \phi$

But $cl(A) \cap cl(B) = \{1\} \neq \phi = cl(A \cap B)$.

(b) (7 points) Suppose A, B, C are three sets. Suppose $f : A \to B$ is surjective. Suppose also that $g : B \to C$ is surjective. True or False: we can conclude that the composition $g \circ f : A \to C$ is always surjective.

TRUE. Take $c \in C$. We aim to find $a \in A$ so that $g \circ f(a) = c$.

Since g surjective, we know there is a $b \in B$ with $g(b) = c$.

Since f surjective, we know there is $a \in A$ with $f(a) = b$.

Hence $g \circ f(a) = g(f(a)) = g(b) = c$ as desired.

(c) (6 points) True or False: The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x + 3$ is injective.

TRUE: To show f injective we show that $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Assume then that $x_1, x_2 \in \mathbb{R}$ with $f(x_1) = f(x_2)$.

THIS Means $5x_1 + 3 = 5x_2 + 3$.

By Algebra, we conclude $5x_1 = 5x_2$ and then $x_1 = x_2$, as desired.

2. (20 points) Are the following two statements True or False? If you believe a statement is False, you must start your answer with the word "False" and then give a counterexample (and explain fully why it is a counterexample). If you believe a statement is True, start your answer with the word "True" and then give a proof.

(a) (7 points) If S is any set, write $\mathbb{P}(S)$ to denote the power set of S - that is , the set of all subsets of S. True or False: For any sets A, B, we have,

$$\mathbb{P}(A \setminus B) = \mathbb{P}(A) \setminus \mathbb{P}(B).$$

FALSE! We can show "$\subseteq$" but not "$\supseteq$". An example is $A = \{1,2,3\}$ and $B = \{1,2\}$

Then $\{1,3\} \in \mathcal{P}(A)$ and $\{1,3\} \notin \mathcal{P}(B)$

so $\{1,3\} \in \mathcal{P}(A) \setminus \mathcal{P}(B)$.

However, $A \setminus B = \{3\}$, so $\{1,3\} \notin \mathcal{P}(A \setminus B)$

(b) (13 points) Recall that if we say a set is 'countable', we mean that the set is either finite, or it is denumerable (in 1-1 correspondance with $\mathbb{N}$). Let $A_1, A_2, \ldots, A_n, \ldots$ be a countable collection of sets and assume each A_i is a finite set. True or False: we can conclude that the countable Cartesian product of these sets is also countable, where this product is defined by,

THIS IS EXACTLY CANTOR'S "DIAGONAL PROCESS"

$$A_1 \times A_2 \times \ldots \times A_n \times \ldots = \{(a_1, a_2, a_3, \ldots) \mid a_i \in A_i \text{ for } i \in \mathbb{N}\}.$$

FALSE! Take e.g. $A_i = \{1,2\}$ for all $i \in \mathbb{N}$

Assume for contradiction that $A_1 \times A_2 \times \cdots$ is countable, and that the elements are $s^{(1)}, s^{(2)}, s^{(3)}, \ldots, s^{(n)}, \ldots$ where $s^{(n)} = (s_1^{(n)}, s_2^{(n)}, s_3^{(n)}, \ldots)$. We define a new sequence $w = (w_1, w_2, \ldots)$ as follows: for all $i \in \mathbb{N}$,

$$w_i = \begin{cases} 1 & \text{if } s_i^{(i)} = 2 \\ 2 & \text{if } s_i^{(i)} = 1 \end{cases}$$

That is, $w_i \neq s_i^{(i)}$ for all i.

By construction $w \neq s^{(n)}$ for any n. But we do have $w \in A_1 \times A_2 \times \cdots \times A_n \times \cdots$. This is a contradiction. So set is uncountable.

3. (10 points) Prove using induction that $9^n - 4^n$ is a multiple of 5 for all $n \in \mathbb{N}$.

We follow the very nice Homework Solution Guide here. (Homework 4)

Base Case: $(n=1)$: We need to show

$9^1 - 4^1 = 5$ is a multiple of 5.

This is immediate.

Inductive Hypothesis: Assume for some $n \in \mathbb{N}$ that we have:

$9^n - 4^n$ is a multiple of 5

We want to show: $9^{(n+1)} - 4^{(n+1)}$ is a multiple of 5.

We calculate:

$$9^{(n+1)} - 4^{(n+1)} = 9^{n+1} - 9 \cdot 4^n + 9 \cdot 4^n - 4 \cdot 4^n$$

$$= 9 \cdot (9^n - 4^n) + (9-4) 4^n$$

$$= 9 \cdot (9^n - 4^n) + 5 \cdot 4^n$$

By Hypothesis, 5 divides $(9^n - 4^n)$. Hence it divides $9 \cdot (9^n - 4^n)$. Also 5 clearly divides the product $5 \cdot 4^n$. Hence 5 divides the sum of these 2 numbers, as desired. $\square$

4. (20 points) For this problem, you are allowed to use the axioms of ordered fields that appear on the cover of this exam **and you are also allowed to assume the following two results about real numbers without proving these two results:**

(I). If $p \in \mathbb{R}$ and $p < 0$ then $p = -q$ for some $q > 0$.

(II). For all $p \in \mathbb{R}, -p = (-1) \cdot p$.

(15 points) Prove the following statement about real numbers using only the assumptions described above: If $x \neq 0$ then $x^2 > 0$. (Here $x^2 = x \cdot x$.)

(Your score on this problem will include a writing score of between 0 and 5 points.)

We follow the very nice Homework solution guide here (Homework 5)

Since $X \neq 0$, we get by O1 that $X < 0$ or $X > 0$.

see cover of exam

Case 1: If $X > 0$, then by O4 we can multiply "$0 < X$" by X to both sides of conclude $0 \cdot X < X \cdot X$,

We know: $0 \cdot X = 0$ since $0 \cdot X \overset{A4}{=} (0+0) \cdot X \overset{DL + M2}{=} 0 \cdot X + 0 \cdot X$

Hence (by A1) This gives $0 \cdot X + (-0X) = (0 \cdot X + 0 \cdot X) + -0 \cdot X$

Hence $0 = 0 \cdot X + (0 \cdot X + -0X) = 0 \cdot X$

By definition $x \cdot x = x^2$

Hence we have shown $0 < x^2$

Case 2: If $X < 0$ then $X = -y$ where $y > 0$ by (I). Hence $X \cdot X = (-y) \cdot (-y) = (-1) y \cdot (-1) y$.

$= (-1)(-1) y \cdot y = -(-y^2) = y^2$ since $y^2 + (-y^2) = 0$

By Case 1, $y^2 > 0$ Hence $x^2 > 0$

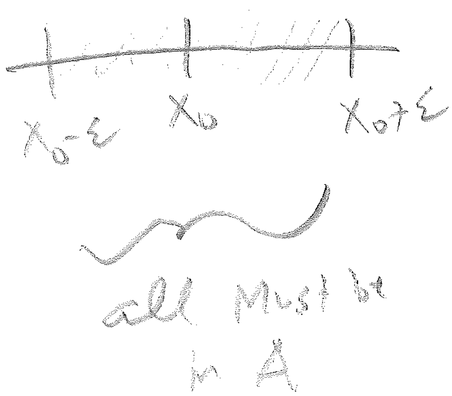
5. (10 points) Is the following statement True or False? If you believe it is False, you must start your answer with the word "False" and then give a counterexample (and explain fully why it is a counterexample). If you believe the statement is True, start your answer with the word "True" and then give a proof.

If $A \subseteq \mathbb{R}$ is nonempty, open and bounded, then we can always conclude that $\sup(A) \notin A$. (The symbol $\notin$ here denotes 'is not an element of'.)

True! Suppose for contradiction that $\sup(A) \in A$. Let's write $x_0 = \sup(A)$. (it exists because A is non-empty and bounded)

If $x_0 \in A$ then x_0 must be an interior point since all points in A are interior points. (A is open) Hence there is an $\varepsilon > 0$ so that $N(x_0, \varepsilon) \subseteq A$. In particular,

$$x_0 + \varepsilon/2 \in A.$$



But we have $x_0 + \varepsilon/2 > x_0$ and this contradicts the fact that x_0 is an upper bound for A.

Hence $\sup(A) \notin A$. $\blacksquare$

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