MATH 3283W
Fall 2017
Exam 3
Thursday 30 November 2017
Time Limit: 50 minutes

Name (Print): ____________________
Student ID: ____________________
Section Number: ______
Teaching Assistant: ____________________
Signature: ____________________

This exam contains 5 numbered problems on seven sheets of paper. The last sheet is blank. Check to see if any pages are missing. Point values are in parentheses. No books, notes, or electronic devices are allowed.

As on the writing quizzes, your work will be graded on the quality of your writing as well as on the validity of the mathematics. In particular, included in the 20 points for problem 3 is a five-point writing score.

Do not use symbols for logical connectives and quantifiers. That is, do not use the symbols $\Rightarrow$, $\Leftrightarrow$, $\wedge$, $\vee$, $\sim$, $\forall$, $\exists$, and $\ni$.

1	20 pts	
2	20 pts	
3	20 pts	
4	10 pts	
5	20 pts	
TOTAL	90 pts	

1. (20 points) Are the following three statements True or False? If you believe a statement is False, you must start your answer with the word "False" and then give a counterexample (and explain why it is a counterexample). If you believe a statement is True, start your answer with the word "True" and then give a proof. (Your proofs can use any result in the textbook reading or any result we covered in the course lectures.)

(a) (7 points)True or False: If the set $S \subset \mathbf{R}$ is compact and x_0 is an accumulation point for S, then $x_0 \in S$.

True. S compact implies that S is closed and bounded.
If x_0 is an accumulation point of S then $x_0 \in S$ because S is closed and closed sets contain all their accumulation points.

(b) (7 points) True or False: Some unbounded sets in $\mathbf{R}$ are compact.

False. If $S \subseteq \mathbb{R}$ is compact it is closed and bounded (by Heine-Borel Theorem). It cannot be unbounded.

(c) (6 points) True or False: If a set has a maximum and a minimum then the set is compact.

False: for example take

$S = [-1, 0) \cup (1, 2]$

Then $\min(S) = -1$
$\max(S) = 2$
But S is not closed
so it is not compact.

2. (20 points) For s_j given by the following formulae, determine the convergence or divergence of the sequence $(s_j)_{j=1}^{\infty}$. Find any limits that exist. (So for example, if the sequence diverges, and also goes to ∞, you should say this.) Show all work and explain your reasoning. You may use any results from the reading or discussed in the lectures.

a) [5 points]

$$s_j = \left(\frac{1}{j}\right)^{\frac{1}{j}}$$

$$s_j = \left(\frac{1}{j}\right)^{\frac{1}{j}} = \frac{1}{(j)^{1/j}}$$

We showed in lecture that $\lim_{j\to\infty} (j)^{1/j} = 1$

So $\lim_{j\to\infty} \frac{1}{(j)^{1/j}} = \frac{1}{\lim_{j\to\infty} (j)^{1/j}} = \frac{1}{1} = 1$

THE sequence converges, and the limit is 1.

b). [7 points]

$$s_j = \frac{j^j}{(2j)!}.$$

$$\lim_{j\to\infty} \frac{s_{j+1}}{s_j} = \lim_{j\to\infty} \frac{(j+1)^{j+1}}{(2(j+1))!} \cdot \frac{(2j)!}{j^j}$$

$$= \lim_{j\to\infty} \frac{(j+1)\cdot(j+1)^{j}}{(2j+2)(2j+1)\cdot j^j} = \lim_{j\to\infty} \left[\frac{j+1}{(2j+2)(2j+1)} \cdot \left(\frac{j+1}{j}\right)^j\right]$$

This goes to zero as $j\to\infty$ because? $= \frac{j(1+\frac{1}{j})}{j^2(4+\frac{6}{j}+\frac{2}{j^2})} = \frac{1}{j}\cdot$ (something going to $\frac{1}{4}$) $\to 0$

$\to e$ by discussion in class as $j\to\infty$

$\boxed{\text{CONVerges!}}$ → So: Limit is zero!!

c) [8 points]

$$s_j = \sqrt{j^2+j} - j$$

See Homework Solution guide!!

$$\sqrt{j^2+j} - j = \sqrt{j^2+j} - j\,\frac{\left(\sqrt{j^2+j}+j\right)}{\left(\sqrt{j^2+j}+j\right)}$$

$$= \frac{j^2+j-j^2}{\sqrt{j^2+j}+j} = \frac{j}{\sqrt{j^2+j}+j}$$

$$= \frac{j}{\sqrt{j^2\left(1+\frac{j}{j^2}\right)}+j}$$

$$= \frac{\cancel{j}}{\cancel{j}\left[\sqrt{1+\frac{1}{j}}+1\right]}$$

$$= \frac{1}{\sqrt{1+\frac{1}{j}}+1} \to \frac{1}{2}$$

as $j \nearrow \infty$

Converges to $\frac{1}{2}$.

3. (20 points) Define a sequence $(s_n)_{n=1}^{\infty}$ as follows: $s_1 = \sqrt{5}$, $s_2 = \sqrt{5+\sqrt{5}}$, $s_3 = \sqrt{5+\sqrt{5+\sqrt{5}}}$, and in general define $s_{n+1} = \sqrt{5+s_n}$. Prove carefully (making it clear and explicit if you use any results discussed in the reading or in the lectures) that the sequence converges and find its limit. The question is worth 15 points, with an additional 5 points assigned as a writing score for this problem.

Clearly $s_n \geq 0$ for all n.

CLAIM: s_n is monotone increasing

($s_{n+1} > s_n$ for all $n \in \mathbb{N}$)

Pf: Induction!!

$n=1$: (Base case)

$s_1 = \sqrt{5}$, $s_2 = \sqrt{5+\sqrt{5}} > \sqrt{5}$ ✓✓

Assume true for $n=k$, for some $k \in \mathbb{N}$

Prove true for $n = k+1$

So: we ~~know~~ assume $s_{k+1} > s_k$

want to show $s_{k+2} > s_{k+1}$

To see this: $s_{k+2} = \sqrt{5+s_{k+1}}$

$> \sqrt{5+s_k} = s_{k+1}$.

by assumption definition. ✓✓

CLAIM: s_n is bounded above by 5.

i.e., we claim

$$S_n < 5 \quad \text{for all } n.$$

Pf: By induction!

n=1: WANT $\sqrt{5} < 5$

This is immediate! ✓

Assume true for $n=k$,

Prove true for $n=k+1$:

$$S_{k+1} = \sqrt{5+S_k} \underset{\substack{\uparrow \\ \text{Induction} \\ \text{Hypothesis}}}{<} \sqrt{5+5} = \sqrt{10}$$

But $\sqrt{10} < 5$ so we are done!! ▣

By Monotone Convergence Theorem

$$\lim_{n\to\infty} S_n = l \quad \text{for some } l.$$

To determine l, look at $S_{n+1} = \sqrt{5+S_n}$ and take $\lim\limits_{n\to\infty}$ of both sides.

Then $$l = \lim_{n\to\infty} \sqrt{5+S_n} \underset{\substack{\uparrow \\ \text{discussed} \\ \text{in reading \& class}}}{=} \sqrt{\lim_{n\to\infty}(5+S_n)} = \sqrt{5+l}$$

Hence $l^2 = 5+l$, Hence $l^2 - l - 5 = 0$

Hence $\boxed{l = \dfrac{1+\sqrt{21}}{2}}$ (only pos. root possible)

4. (10 points) Determine the limit of the sequence (s_n) with terms,

$$s_n = \frac{2n+3}{n^2-13}.$$

Justify your answer directly from the definition of convergence. Do not use any theorems that have been proven in class or in the textbook.

We first use our intuition: It seems like s_n will behave like $\frac{2n}{n^2} \sim \frac{2}{n} \to 0$

Let's show this carefully!!

Given $\varepsilon > 0$, WANT to find N so $n \geq N$ implies

$$\underbrace{|s_n - 0|}_{} < \varepsilon.$$

Calculate: $|s_n - 0| = \left| \frac{2n+3}{n^2-13} \right|$

$= \frac{2n+3}{n^2-13}$ ← If $n \geq 4$, numerator & denominator are positive!

Now note: if $n \geq 6$ then $n > 3$ and $13 < \frac{1}{2}n^2$

So for $n \geq 6$ we can continue as follows:

$$\frac{2n+3}{n^2-13} < \frac{2n+n}{n^2 - \frac{1}{2}n^2} = \frac{3n}{\frac{1}{2}n^2} = \frac{6}{n}$$

So: Choose $N > \max\{6, \frac{6}{\varepsilon}\}$ then $n \geq N$ →

implies

$$|s_n - 0| < \frac{6}{n} < \frac{6}{\frac{6}{\varepsilon}} = \varepsilon$$

as desired.

5. (20 points) (a) (3 points) Given any bounded sequence $(u_n)_{n=1}^{\infty}$, define what we mean by $\liminf u_n$ and $\limsup u_n$.

Let S be the set of all subsequential limits of (u_n). Then $\liminf u_n = \inf S$ and $\limsup u_n = \sup S$.

(b). (2 points) Consider the sequence $(u_n)_{n=1}^{\infty}$ where,

$$u_n = n^2(-1 + (-1)^n),$$

and state what values the two quantities $\liminf u_n$ and $\limsup u_n$ have for this particular example.

u_n looks like: $\begin{cases} 0 & \text{if } n \text{ even} \\ -2n^2 & \text{if } n \text{ odd.} \end{cases}$ ← These terms go to $-\infty$, and there are no strictly positive terms, and even terms converge to 0

$\liminf u_n = -\infty$ ←

$\limsup u_n = 0$

(c) (15 points). Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers which is bounded. Let $L = \limsup a_n$. Prove that there is a subsequence $(a_{n_j})_{j=1}^{\infty}$ so that $\lim_{j\to\infty} a_{n_j} = L$. You may use any result discussed in class or the reading in our textbook in your proof.

We discussed in class that if we define:

$$B_k = \sup\{a_k, a_{k+1}, a_{k+2}, \dots\}$$

Then $B_k \geq B_{k+1} \geq \dots$

and $L = \lim_{k\to\infty} B_k$.

Pick n_1 as small as possible so that $(B_1 - a_{n_1}) < \frac{1}{2}$

Next, pick n_2 " " " " so $n_2 \geq n_1$ and $B_2 - a_{n_2} < \frac{1}{2^2} = \frac{1}{4}$

Continue in this way: pick $n_k \geq n_{k-1}$ so that n_k is as small as possible, & $(B_k - a_{n_k}) < \frac{1}{2^k}$.

⟶

⟶

We know that

$$\lim_{n\to\infty} B_n = L = \limsup a_n$$

Given any $\varepsilon > 0$, Pick N large enough so that 2 condition are met:

① $|B_n - L| < \varepsilon/2$ for all $n \geq N$

② $2^{-N} < \varepsilon/2$

Then for all $k \geq N$ we have:

$$\begin{aligned} |a_{n_k} - L| &\leq |a_{n_k} - B_k| + |B_k - L| \\ &< 2^{-k} + \varepsilon/2 \\ &< 2^{-N} + \varepsilon/2 \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

As desired!

Here is a second proof of the result!

Let S be the set of all subsequential limits of $(a_n)_{n=1}^{\infty}$

Then $L = \sup S$.

Hence there is a subsequential limit $l_1 \in S$ so $|L - l_1| < \frac{1}{2}$.

Pick $n_1 \in \mathbb{N}$ to be smallest possible so that $|a_{n_1} - l_1| < \frac{1}{2}$.

Similarly, if we have defined $n_1, n_2, n_3, \dots, n_k$)
we define n_{k+1} as follows:

There is a subsequential limit $l_{k+1} \in S$ so that $|L - l_{k+1}| < \frac{1}{2(k+1)}$

and define n_{k+1} to be smallest possible in $\mathbb{N}$ so that $n_{k+1} > n_k$ and $|a_{n_{k+1}} - l_{k+1}| < \frac{1}{2(k+1)}$

CLAIM: $\lim_{k\to\infty} a_{n_k} = L$. Proof: Given $\varepsilon > 0$, pick N large enough so $\frac{1}{N} < \varepsilon$. Then $k > N$ gives:

$$|a_{n_k} - L| \le |a_{n_k} - l_k| + |l_k - L| \le \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} < \frac{1}{N} < \varepsilon.$$

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