1. (20 points) Are the following three statements True or False? If you believe a statement is False, you must start your answer with the word "False" and then give a counterexample (and explain why it is a counterexample). If you believe a statement is True, start your answer with the word "True" and then give a proof. (Your proofs can use any result in the textbook reading or any result we covered in the course lectures.)

(a) (5 points)True or False: The union of any collection of closed sets in $\mathbb{R}$ is also a closed set.

False. The union of any finite collection of closed sets is closed.

For a counterexample, consider $A_n = [\frac{1}{n}, 1-\frac{1}{n}]$ for $n \in \mathbb{N}, n \geq 2$.
Each set A_n is closed, but $\bigcup_{n=2}^{\infty} A_n = (0,1)$ is not closed.

(b) (5 points) True or False: Given sequences (s_n), (t_n) with $s_n \leq t_n$ for all $n \in \mathbb{N}$, if $\lim s_n = \infty$ then $\lim t_n = \infty$.

True. Since $\lim_{n\to\infty} s_n = \infty$, for every $M \in \mathbb{R}$ there exists $N_0 \in \mathbb{N}$ such that for $n > N_0$ we know that $s_n > M$. Since $t_n \geq s_n$ for all n, this also means that for $n > N_0$, $t_n > M$. Thus $\lim t_n = \infty$.

(c). (10 points) True or False: If S is a nonempty bounded set, and if we set $m = \sup S$, then m is a boundary point of the set S.

True.

To prove that $m = \sup S$ is a boundary point of S, we must show that for every $\varepsilon > 0$, $N(m,\varepsilon) \cap S \neq \emptyset$ and $N(m,\varepsilon) \cap S^c \neq \emptyset$.

Given $\varepsilon > 0$, since m is the supremum of S, there exists $s \in S$ such that $m - \varepsilon < s \leq m$. This means $s \in N(m,\varepsilon)$, so $N(m,\varepsilon) \cap S \neq \emptyset$. Also, since $m = \sup S$, we know that $y \notin S$ for all $y > m$. That is, $y \in S^c$ for all $y > m$. Consider $y = m + \frac{\varepsilon}{2}$. Then $y > m$ so $y \in S^c$ and also $y \in N(m,\varepsilon)$. Thus $N(m,\varepsilon) \cap S^c \neq \emptyset$. Therefore $m = \sup S$ is a boundary point of S.

2. (15 points) Suppose S is a compact subset of the real line $\mathbb{R}$. Prove that any infinite subset of S has an accumulation point in S. You may use any result proved or discussed in the textbook reading.

Suppose T is an infinite subset of S.

Since S is compact, Heine-Borel implies that S is closed and bounded.

Since T is a subset of S, T is also bounded.

By Bolzano-Weierstrass, T then has an accumulation point $x_0 \in \mathbb{R}$ and since $T \subseteq S$, x_0 is also an accumulation point of S.

S contains all of it's accumulation points because it is closed, so $x_0 \in S$.

Thus T has an accumulation point in S.

3. (15 points) Determine whether the following series converges or diverges and justify your answer. (You may use any test we discussed in class or in the reading, and you may state without proof the convergence or divergence of any series we discussed in class or which was discussed in the reading.)

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}.$$

First, we note that $0 < \frac{1}{n \cdot \sqrt{n+1}}$ for $n \in \mathbb{N}$ and for all $n \in \mathbb{N}$ $\frac{1}{n \cdot \sqrt{n+1}} < \frac{1}{n \cdot \sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}$.

Now, as we know $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges (it is p-series with $p = \frac{3}{2} > 1$).

Finally, since $0 < \frac{1}{n \cdot \sqrt{n+1}} < \frac{1}{n^{\frac{3}{2}}}$ and ~~$\sum n^{\frac{3}{2}}$~~ $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges by the comparision test $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}$ also converges.

4. Suppose that (s_n) is a convergent sequence with $\lim_{n\to\infty} s_n = s$. Suppose also that $k > 0$ is a constant.

Prove, using only the definition of the limit, that

$$\lim_{n\to\infty} (k \cdot s_n) = k \cdot s.$$

<u>Proof</u>: Let $\varepsilon > 0$ be given. Because $\lim_{n\to\infty} s_n = s$, there exists some $N \in \mathbb{N}$ such that $n > N$ implies $|s_n - s| < \frac{\varepsilon}{k}$. Thus $n > N$ implies

$$|ks_n - k \cdot s| = |k|\,|s_n - s| = k\,|s_n - s| < k \cdot \frac{\varepsilon}{k} = \varepsilon.$$

Thus for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $n > N$ implies $|ks_n - ks| < \varepsilon$. By the definition of the limit, it follows that $\lim_{n\to\infty} ks_n = k \cdot s$.

$\square$

#5 Define $\{s_n\}_{n=1}^{\infty}$ by $s_1 = 2$ and $s_{n+1} = \frac{1}{5}(3s_n + 10)$ for $n \geq 1$. Prove that the limit of the sequence exists and find what it is equal to.

Let $\{s_n\}$ be as stated. We first show that $\{s_n\}$ is increasing. Note that $s_1 = 2$ and $s_2 = \frac{16}{5}$. Since $2 < 16/5$ s_n is increasing from s_1 to s_2. Now, suppose that $s_n < s_{n+1}$ for some $n \in \mathbb{N}$. Then,

$$s_{n+2} = \frac{1}{5}(3 \cdot s_{n+1} + 10) > \frac{1}{5}(3 \cdot s_n + 10) = s_{n+1}.$$

Thus, $\{s_n\}$ is increasing.

We now show that $\{s_n\}$ is bounded. Since $\{s_n\}$ is increasing, it is bounded from below by $s_1 = 2$. Consider $m = 20$. Note that $s_1 = 2 < 20$. So, suppose $s_n < 20$ for some $n \in \mathbb{N}$. Then,

$$s_{n+1} = \frac{1}{5}(3 \cdot s_n + 10) < \frac{1}{5}(3 \cdot 20 + 10) = \frac{1}{5}(70) = 70/5 < 20.$$

Thus, $\{s_n\}$ is bounded from above by $m = 20$.

Since $\{s_n\}$ is bounded and monotone, the Monotone Convergence Thm implies that $\lim_{n\to\infty} s_n = s$ exists. Since $\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} s_n = s$, we have $\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \frac{1}{5}(3 s_n + 10)$. Thus, s solves $s = \frac{1}{5}(3 \cdot s + 10)$. Simple algebra shows $s = 5$.

6. (15 points) .

(a).(5 points) Consider the sequence $(u_n)_{n=1}^{\infty}$ where,

$$u_n = \sin(n) + (-n)^n.$$

What are liminf u_n and limsup u_n ? Justify your answer briefly.

Since for each $n \in \mathbb{N}$, $|\sin(n)| \leq 1$ $|(-n)^n| = n^n$, it follows that $\{u_n\}$ is unbounded both above and below, since $(-n)^n$ alternates in sign.
Hence $\liminf u_n = -\infty$, $\limsup u_n = +\infty$,

(b). (10 points). Let $(s_n), (t_n)$ both be bounded sequences in R. Prove that,

$$\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n.$$

Let $s := \limsup s_n$, $t := \limsup t_n$.
Then for each $\varepsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that for all $n \geq N_1$, $s_n < s + \frac{\varepsilon}{2}$, and for all $n \geq N_2$, $t_n < t + \frac{\varepsilon}{2}$.
Let $N := \max\{N_1, N_2\}$. Then for each $n \geq N$, we have

$$s_n + t_n < (s + \tfrac{\varepsilon}{2}) + (t + \tfrac{\varepsilon}{2}) = s + t + \varepsilon.$$

In particular,

$$\limsup(s_n + t_n) < s + t + \varepsilon, \quad \text{for each } \varepsilon > 0.$$

Taking $\varepsilon \downarrow 0$ in the last expression yields

$$\limsup(s_n + t_n) \leq s + t,$$

as desired.

$\square$