Solutions for Chapter 3

Solutions for exercises in section 3. 2

3.2.1. (a) $\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$ (b) $x = -\frac{1}{2}$, $y = -6$, and $z = 0$

3.2.2. (a) Neither (b) Skew symmetric (c) Symmetric (d) Neither

3.2.3. The 3×3 zero matrix trivially satisfies all conditions, and it is the only possible answer for part (a). The only possible answers for (b) are real symmetric matrices. There are many nontrivial possibilities for (c).

3.2.4. $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{B} = \mathbf{B}^T \implies (\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T = \mathbf{A} + \mathbf{B}$. Yes—the skew-symmetric matrices are also closed under matrix addition.

3.2.5. (a) $\mathbf{A} = -\mathbf{A}^T \implies a_{ij} = -a_{ji}$. If $i = j$, then $a_{jj} = -a_{jj} \implies a_{jj} = 0$.

(b) $\mathbf{A} = -\mathbf{A}^* \implies a_{ij} = -\overline{a_{ji}}$. If $i = j$, then $a_{jj} = -\overline{a_{jj}}$. Write $a_{jj} = x + \mathrm{i}y$ to see that $a_{jj} = -\overline{a_{jj}} \implies x + \mathrm{i}y = -x + \mathrm{i}y \implies x = 0 \implies a_{jj}$ is pure imaginary.

(c) $\mathbf{B}^* = (\mathrm{i}\mathbf{A})^* = -\mathrm{i}\mathbf{A}^* = -\mathrm{i}\overline{\mathbf{A}}^T = -\mathrm{i}\mathbf{A}^T = -\mathrm{i}\mathbf{A} = -\mathbf{B}$.

3.2.6. (a) Let $\mathbf{S} = \mathbf{A} + \mathbf{A}^T$ and $\mathbf{K} = \mathbf{A} - \mathbf{A}^T$. Then $\mathbf{S}^T = \mathbf{A}^T + \mathbf{A}^{T^T} = \mathbf{A}^T + \mathbf{A} = \mathbf{S}$. Likewise, $\mathbf{K}^T = \mathbf{A}^T - \mathbf{A}^{T^T} = \mathbf{A}^T - \mathbf{A} = -\mathbf{K}$.

(b) $\mathbf{A} = \frac{\mathbf{S}}{2} + \frac{\mathbf{K}}{2}$ is one such decomposition. To see it is unique, suppose $\mathbf{A} = \mathbf{X} + \mathbf{Y}$, where $\mathbf{X} = \mathbf{X}^T$ and $\mathbf{Y} = -\mathbf{Y}^T$. Thus, $\mathbf{A}^T = \mathbf{X}^T + \mathbf{Y}^T = \mathbf{X} - \mathbf{Y} \implies \mathbf{A} + \mathbf{A}^T = 2\mathbf{X}$, so that $\mathbf{X} = \frac{\mathbf{A}+\mathbf{A}^T}{2} = \frac{\mathbf{S}}{2}$. A similar argument shows that $\mathbf{Y} = \frac{\mathbf{A}-\mathbf{A}^T}{2} = \frac{\mathbf{K}}{2}$.

3.2.7. (a) $[(\mathbf{A}+\mathbf{B})^*]_{ij} = [\overline{\mathbf{A}+\mathbf{B}}]_{ji} = [\overline{\mathbf{A}} + \overline{\mathbf{B}}]_{ji} = [\overline{\mathbf{A}}]_{ji} + [\overline{\mathbf{B}}]_{ji} = [\mathbf{A}^*]_{ij} + [\mathbf{B}^*]_{ij} = [\mathbf{A}^* + \mathbf{B}^*]_{ij}$

(b) $[(\alpha\mathbf{A})^*]_{ij} = [\overline{\alpha\mathbf{A}}]_{ji} = [\bar{\alpha}\overline{\mathbf{A}}]_{ji} = \bar{\alpha}[\overline{\mathbf{A}}]_{ji} = \bar{\alpha}[\mathbf{A}^*]_{ij}$

3.2.8. $$k \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$$

Solutions for exercises in section 3. 3

3.3.1. Functions (b) and (f) are linear. For example, to check if (b) is linear, let $\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, and check if $f(\mathbf{A}+\mathbf{B}) = f(\mathbf{A}) + f(\mathbf{B})$ and

$f(\alpha\mathbf{A}) = \alpha f(\mathbf{A})$. Do so by writing

$$f(\mathbf{A}+\mathbf{B}) = f\begin{pmatrix} a_1+b_1 \\ a_2+b_2 \end{pmatrix} = \begin{pmatrix} a_2+b_2 \\ a_1+b_1 \end{pmatrix} = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} + \begin{pmatrix} b_2 \\ b_1 \end{pmatrix} = f(\mathbf{A}) + f(\mathbf{B}),$$

$$f(\alpha\mathbf{A}) = f\begin{pmatrix} \alpha a_1 \\ \alpha a_2 \end{pmatrix} = \begin{pmatrix} \alpha a_2 \\ \alpha a_1 \end{pmatrix} = \alpha\begin{pmatrix} a_2 \\ a_1 \end{pmatrix} = \alpha f(\mathbf{A}).$$

3.3.2. Write $f(\mathbf{x}) = \sum_{i=1}^{n} \xi_i x_i$. For all points $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$, and for all scalars α, it is true that

$$\begin{aligned} f(\alpha\mathbf{x}+\mathbf{y}) &= \sum_{i=1}^{n} \xi_i(\alpha x_i + y_i) = \sum_{i=1}^{n} \xi_i \alpha x_i + \sum_{i=1}^{n} \xi_i y_i \\ &= \alpha\sum_{i=1}^{n} \xi_i x_i + \sum_{i=1}^{n} \xi_i y_i = \alpha f(\mathbf{x}) + f(\mathbf{y}). \end{aligned}$$

3.3.3. There are many possibilities. Two of the simplest and most common are Hooke's law for springs that says that $F = kx$ (see Example 3.2.1) and Newton's second law that says that $F = ma$ (i.e., force = mass × acceleration).

3.3.4. They are all linear. To see that rotation is linear, use trigonometry to deduce that if $\mathbf{p} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, then $f(\mathbf{p}) = \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, where

$$\begin{aligned} u_1 &= (\cos\theta)x_1 - (\sin\theta)x_2 \\ u_2 &= (\sin\theta)x_1 + (\cos\theta)x_2. \end{aligned}$$

f is linear because this is a special case of Example 3.3.2. To see that reflection is linear, write $\mathbf{p} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $f(\mathbf{p}) = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$. Verification of linearity is straightforward. For the projection function, use the Pythagorean theorem to conclude that if $\mathbf{p} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, then $f(\mathbf{p}) = \frac{x_1+x_2}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Linearity is now easily verified.

Solutions for exercises in section 3. 4

3.4.1. Refer to the solution for Exercise 3.3.4. If $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{P}$ denote the matrices associated with the rotation, reflection, and projection, respectively, then

$$\mathbf{Q}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix},\quad \mathbf{R}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \text{and}\quad \mathbf{P}=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}.$$

3.4.2. Refer to the solution for Exercise 3.4.1 and write

$$\mathbf{RQ}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ -\sin\theta & -\cos\theta\end{pmatrix}.$$

If $Q(\mathbf{x})$ is the rotation function and $R(\mathbf{x})$ is the reflection function, then the composition is

$$R\big(Q(\mathbf{x})\big)=\begin{pmatrix}(\cos\theta)x_1-(\sin\theta)x_2\\ -(\sin\theta)x_1-(\cos\theta)x_2\end{pmatrix}.$$

3.4.3. Refer to the solution for Exercise 3.4.1 and write

$$\begin{aligned}\mathbf{PQR}&=\begin{pmatrix}a_{11}x_1+a_{12}x_2\\ a_{21}x_1+a_{22}x_2\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\ &=\frac{1}{2}\begin{pmatrix}\cos\theta+\sin\theta & \sin\theta-\cos\theta\\ \cos\theta+\sin\theta & \sin\theta-\cos\theta\end{pmatrix}.\end{aligned}$$

Therefore, the composition of the three functions in the order asked for is

$$P\Big(Q\big(R(\mathbf{x})\big)\Big)=\frac{1}{2}\begin{pmatrix}(\cos\theta+\sin\theta)x_1+(\sin\theta-\cos\theta)x_2\\ (\cos\theta+\sin\theta)x_1+(\sin\theta-\cos\theta)x_2\end{pmatrix}.$$

Solutions for exercises in section 3. 5

3.5.1. (a) $\mathbf{AB}=\begin{pmatrix}10 & 15\\ 12 & 8\\ 28 & 52\end{pmatrix}$ (b) $\mathbf{BA}$ does not exist (c) $\mathbf{CB}$ does not exist

(d) $\mathbf{C}^T\mathbf{B}=\begin{pmatrix}10 & 31\end{pmatrix}$ (e) $\mathbf{A}^2=\begin{pmatrix}13 & -1 & 19\\ 16 & 13 & 12\\ 36 & -17 & 64\end{pmatrix}$ (f) $\mathbf{B}^2$ does not exist

(g) $\mathbf{C}^T\mathbf{C}=14$ (h) $\mathbf{CC}^T=\begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & 6 & 9\end{pmatrix}$ (i) $\mathbf{BB}^T=\begin{pmatrix}5 & 8 & 17\\ 8 & 16 & 28\\ 17 & 28 & 58\end{pmatrix}$

(j) $\mathbf{B}^T\mathbf{B}=\begin{pmatrix}10 & 23\\ 23 & 69\end{pmatrix}$ (k) $\mathbf{C}^T\mathbf{AC}=76$

3.5.2. (a) $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 10 \\ -2 \end{pmatrix}$ (b) $\mathbf{s} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

(c) $\mathbf{b} = \mathbf{A}_{*1} - 2\mathbf{A}_{*2} + 3\mathbf{A}_{*3}$

3.5.3. (a) $\mathbf{EA} = \begin{pmatrix} \mathbf{A}_{1*} \\ \mathbf{A}_{2*} \\ 3\mathbf{A}_{1*} + \mathbf{A}_{3*} \end{pmatrix}$ (b) $\mathbf{AE} = \begin{pmatrix} \mathbf{A}_{*1} + 3\mathbf{A}_{*3} & \mathbf{A}_{*2} & \mathbf{A}_{*3} \end{pmatrix}$

3.5.4. (a) $\mathbf{A}_{*j}$ (b) $\mathbf{A}_{i*}$ (c) a_{ij}

3.5.5. $\mathbf{Ax} = \mathbf{Bx}\ \forall\ \mathbf{x} \implies \mathbf{Ae}_j = \mathbf{Be}_j\ \forall\ \mathbf{e}_j \implies \mathbf{A}_{*j} = \mathbf{B}_{*j}\ \forall\ j \implies \mathbf{A} = \mathbf{B}$. (The symbol $\forall$ is mathematical shorthand for the phrase "for all.")

3.5.6. The limit is the zero matrix.

3.5.7. If $\mathbf{A}$ is $m \times p$ and $\mathbf{B}$ is $p \times n$, write the product as

$$\mathbf{AB} = \begin{pmatrix} \mathbf{A}_{*1} & \mathbf{A}_{*2} & \cdots & \mathbf{A}_{*p} \end{pmatrix} \begin{pmatrix} \mathbf{B}_{1*} \\ \mathbf{B}_{2*} \\ \vdots \\ \mathbf{B}_{p*} \end{pmatrix} = \mathbf{A}_{*1}\mathbf{B}_{1*} + \mathbf{A}_{*2}\mathbf{B}_{2*} + \cdots + \mathbf{A}_{*p}\mathbf{B}_{p*}$$

$$= \sum_{k=1}^{p} \mathbf{A}_{*k}\mathbf{B}_{k*}.$$

3.5.8. (a) $[\mathbf{AB}]_{ij} = \mathbf{A}_{i*}\mathbf{B}_{*j} = \begin{pmatrix} 0 & \cdots & 0 & a_{ii} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ \vdots \\ b_{jj} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is 0 when $i > j$.

(b) When $i = j$, the only nonzero term in the product $\mathbf{A}_{i*}\mathbf{B}_{*i}$ is $a_{ii}b_{ii}$.

(c) Yes.

3.5.9. Use $[\mathbf{AB}]_{ij} = \sum_k a_{ik}b_{kj}$ along with the rules of differentiation to write

$$\frac{d[\mathbf{AB}]_{ij}}{dt} = \frac{d\left(\sum_k a_{ik}b_{kj}\right)}{dt} = \sum_k \frac{d(a_{ik}b_{kj})}{dt}$$

$$= \sum_k \left(\frac{da_{ik}}{dt} b_{kj} + a_{ik}\frac{db_{kj}}{dt} \right) = \sum_k \frac{da_{ik}}{dt} b_{kj} + \sum_k a_{ik}\frac{db_{kj}}{dt}$$

$$= \left[\frac{d\mathbf{A}}{dt}\mathbf{B}\right]_{ij} + \left[\mathbf{A}\frac{d\mathbf{B}}{dt}\right]_{ij} = \left[\frac{d\mathbf{A}}{dt}\mathbf{B} + \mathbf{A}\frac{d\mathbf{B}}{dt}\right]_{ij}.$$

3.5.10. (a) $[\mathbf{Ce}]_i =$ the total number of paths *leaving* node i.

(b) $[\mathbf{e}^T\mathbf{C}]_i =$ the total number of paths *entering* node i.

3.5.11. At time t, the concentration of salt in tank i is $\frac{x_i(t)}{V}$ lbs/gal. For tank 1,

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{\text{lbs}}{\text{sec}} \text{ coming in} - \frac{\text{lbs}}{\text{sec}} \text{ going out} = 0\frac{\text{lbs}}{\text{sec}} - \left(r\frac{\text{gal}}{\text{sec}} \times \frac{x_1(t)}{V}\frac{\text{lbs}}{\text{gal}}\right) \\ &= -\frac{r}{V}x_1(t)\frac{\text{lbs}}{\text{sec}}.\end{aligned}$$

For tank 2,

$$\begin{aligned}\frac{dx_2}{dt} &= \frac{\text{lbs}}{\text{sec}} \text{ coming in} - \frac{\text{lbs}}{\text{sec}} \text{ going out} = \frac{r}{V}x_1(t)\frac{\text{lbs}}{\text{sec}} - \left(r\frac{\text{gal}}{\text{sec}} \times \frac{x_2(t)}{V}\frac{\text{lbs}}{\text{gal}}\right) \\ &= \frac{r}{V}x_1(t)\frac{\text{lbs}}{\text{sec}} - \frac{r}{V}x_2(t)\frac{\text{lbs}}{\text{sec}} = \frac{r}{V}\Big(x_1(t) - x_2(t)\Big),\end{aligned}$$

and for tank 3,

$$\begin{aligned}\frac{dx_3}{dt} &= \frac{\text{lbs}}{\text{sec}} \text{ coming in} - \frac{\text{lbs}}{\text{sec}} \text{ going out} = \frac{r}{V}x_2(t)\frac{\text{lbs}}{\text{sec}} - \left(r\frac{\text{gal}}{\text{sec}} \times \frac{x_3(t)}{V}\frac{\text{lbs}}{\text{gal}}\right) \\ &= \frac{r}{V}x_2(t)\frac{\text{lbs}}{\text{sec}} - \frac{r}{V}x_3(t)\frac{\text{lbs}}{\text{sec}} = \frac{r}{V}\Big(x_2(t) - x_3(t)\Big).\end{aligned}$$

This is a system of three linear first-order differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= \tfrac{r}{V}\Big(-x_1(t) \Big) \\ \frac{dx_2}{dt} &= \tfrac{r}{V}\Big(x_1(t) - x_2(t) \Big) \\ \frac{dx_3}{dt} &= \tfrac{r}{V}\Big(x_2(t) - x_3(t) \Big)\end{aligned}$$

that can be written as a single matrix differential equation

$$\begin{pmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{pmatrix} = \frac{r}{V}\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

Solutions for exercises in section 3. 6

3.6.1.

$$\mathbf{AB} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \end{pmatrix} \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11}\mathbf{B}_1 + \mathbf{A}_{12}\mathbf{B}_2 + \mathbf{A}_{13}\mathbf{B}_3 \\ \mathbf{A}_{21}\mathbf{B}_1 + \mathbf{A}_{22}\mathbf{B}_2 + \mathbf{A}_{23}\mathbf{B}_3 \end{pmatrix}$$

$$= \left(\begin{array}{cc} -10 & -19 \\ -10 & -19 \\ \hline -1 & -1 \end{array}\right)$$

3.6.2. Use block multiplication to verify $\mathbf{L}^2 = \mathbf{I}$—be careful not to commute any of the terms when forming the various products.

3.6.3. Partition the matrix as $\mathbf{A} = \begin{pmatrix} \mathbf{I} & \mathbf{C} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}$, where $\mathbf{C} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and observe that $\mathbf{C}^2 = \mathbf{C}$. Use this together with block multiplication to conclude that

$$\mathbf{A}^k = \begin{pmatrix} \mathbf{I} & \mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3 + \cdots + \mathbf{C}^k \\ \mathbf{0} & \mathbf{C}^k \end{pmatrix} = \begin{pmatrix} \mathbf{I} & k\mathbf{C} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}.$$

Therefore, $\mathbf{A}^{300} = \begin{pmatrix} 1 & 0 & 0 & 100 & 100 & 100 \\ 0 & 1 & 0 & 100 & 100 & 100 \\ 0 & 0 & 1 & 100 & 100 & 100 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{pmatrix}$.

3.6.4. $(\mathbf{A}^*\mathbf{A})^* = \mathbf{A}^*\mathbf{A}^{**} = \mathbf{A}^*\mathbf{A}$ and $(\mathbf{AA}^*)^* = \mathbf{A}^{**}\mathbf{A}^* = \mathbf{AA}^*$.

3.6.5. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T = \mathbf{BA} = \mathbf{AB}$. It is easy to construct a 2×2 example to show that this need not be true when $\mathbf{AB} \neq \mathbf{BA}$.

3.6.6.

$$\begin{aligned} [(\mathbf{D}+\mathbf{E})\mathbf{F}]_{ij} &= (\mathbf{D}+\mathbf{E})_{i*}\mathbf{F}_{*j} = \sum_k [\mathbf{D}+\mathbf{E}]_{ik}[\mathbf{F}]_{kj} = \sum_k \left([\mathbf{D}]_{ik} + [\mathbf{E}]_{ik}\right)[\mathbf{F}]_{kj} \\ &= \sum_k \left([\mathbf{D}]_{ik}[\mathbf{F}]_{kj} + [\mathbf{E}]_{ik}[\mathbf{F}]_{kj}\right) = \sum_k [\mathbf{D}]_{ik}[\mathbf{F}]_{kj} + \sum_k [\mathbf{E}]_{ik}[\mathbf{F}]_{kj} \\ &= \mathbf{D}_{i*}\mathbf{F}_{*j} + \mathbf{E}_{i*}\mathbf{F}_{*j} = [\mathbf{DF}]_{ij} + [\mathbf{EF}]_{ij} \\ &= [\mathbf{DF} + \mathbf{EF}]_{ij}. \end{aligned}$$

3.6.7. If a matrix $\mathbf{X}$ did indeed exist, then

$$\begin{aligned} \mathbf{I} = \mathbf{AX} - \mathbf{XA} \quad &\Longrightarrow \quad trace\,(\mathbf{I}) = trace\,(\mathbf{AX} - \mathbf{XA}) \\ &\Longrightarrow \quad n = trace\,(\mathbf{AX}) - trace\,(\mathbf{XA}) = 0, \end{aligned}$$

which is impossible.

3.6.8. (a) $\mathbf{y}^T\mathbf{A} = \mathbf{b}^T \implies (\mathbf{y}^T\mathbf{A})^T = \mathbf{b}^{TT} \implies \mathbf{A}^T\mathbf{y} = \mathbf{b}$. This is an $n \times m$ system of equations whose coefficient matrix is $\mathbf{A}^T$. (b) They are the same.

3.6.9. Draw a transition diagram similar to that in Example 3.6.3 with North and South replaced by ON and OFF, respectively. Let x_k be the proportion of switches in the ON state, and let y_k be the proportion of switches in the OFF state after k clock cycles have elapsed. According to the given information,

$$\begin{aligned} x_k &= x_{k-1}(.1) + y_{k-1}(.3) \\ y_k &= x_{k-1}(.9) + y_{k-1}(.7) \end{aligned}$$

so that $\mathbf{p}_k = \mathbf{p}_{k-1}\mathbf{P}$, where

$$\mathbf{p}_k = \begin{pmatrix} x_k & y_k \end{pmatrix} \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} .1 & .9 \\ .3 & .7 \end{pmatrix}.$$

Just as in Example 3.6.3, $\mathbf{p}_k = \mathbf{p}_0\mathbf{P}^k$. Compute a few powers of $\mathbf{P}$ to find

$$\mathbf{P}^2 = \begin{pmatrix} .280 & .720 \\ .240 & .760 \end{pmatrix}, \quad \mathbf{P}^3 = \begin{pmatrix} .244 & .756 \\ .252 & .748 \end{pmatrix}$$

$$\mathbf{P}^4 = \begin{pmatrix} .251 & .749 \\ .250 & .750 \end{pmatrix}, \quad \mathbf{P}^5 = \begin{pmatrix} .250 & .750 \\ .250 & .750 \end{pmatrix}$$

and deduce that $\mathbf{P}^\infty = \lim_{k\to\infty} \mathbf{P}^k = \begin{pmatrix} 1/4 & 3/4 \\ 1/4 & 3/4 \end{pmatrix}$. Thus

$$\mathbf{p}_k \to \mathbf{p}_0\mathbf{P}^\infty = \begin{pmatrix} \frac{1}{4}(x_0+y_0) & \frac{3}{4}(x_0+y_0) \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

For practical purposes, the device can be considered to be in equilibrium after about 5 clock cycles—regardless of the initial proportions.

3.6.10. $\begin{pmatrix} -4 & 1 & -6 & 5 \end{pmatrix}$

3.6.11. (a) $\mathit{trace}\,(\mathbf{ABC}) = \mathit{trace}\,(\mathbf{A}\{\mathbf{BC}\}) = \mathit{trace}\,(\{\mathbf{BC}\}\mathbf{A}) = \mathit{trace}\,(\mathbf{BCA})$. The other equality is similar. (b) Use almost any set of 2×2 matrices to construct an example that shows equality need not hold. (c) Use the fact that $\mathit{trace}\left(\mathbf{C}^T\right) = \mathit{trace}\,(\mathbf{C})$ for all square matrices to conclude that

$$\begin{aligned} \mathit{trace}\left(\mathbf{A}^T\mathbf{B}\right) &= \mathit{trace}\left(\left(\mathbf{A}^T\mathbf{B}\right)^T\right) = \mathit{trace}\left(\mathbf{B}^T\mathbf{A}^{T^T}\right) \\ &= \mathit{trace}\left(\mathbf{B}^T\mathbf{A}\right) = \mathit{trace}\left(\mathbf{A}\mathbf{B}^T\right). \end{aligned}$$

3.6.12. (a) $\mathbf{x}^T\mathbf{x} = 0 \iff \sum_{k=1}^n x_i^2 = 0 \iff x_i = 0$ for each $i \iff \mathbf{x} = \mathbf{0}$.

(b)
$$\begin{aligned} \mathit{trace}\left(\mathbf{A}^T\mathbf{A}\right) = 0 &\iff \sum_i [\mathbf{A}^T\mathbf{A}]_{ii} = 0 \iff \sum_i (\mathbf{A}^T)_{i*}\mathbf{A}_{*i} = 0 \\ &\iff \sum_i \sum_k [\mathbf{A}^T]_{ik}[\mathbf{A}]_{ki} = 0 \iff \sum_i \sum_k [\mathbf{A}]_{ki}[\mathbf{A}]_{ki} = 0 \\ &\iff \sum_i \sum_k [\mathbf{A}]_{ki}^2 = 0 \\ &\iff [\mathbf{A}]_{ki} = 0 \text{ for each } k \text{ and } i \iff \mathbf{A} = \mathbf{0} \end{aligned}$$

Solutions for exercises in section 3. 7

3.7.1. (a) $\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$ (b) Singular (c) $\begin{pmatrix} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -8 & 4 \end{pmatrix}$ (d) Singular

(e) $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$

3.7.2. Write the equation as $(\mathbf{I}-\mathbf{A})\mathbf{X}=\mathbf{B}$ and compute

$$\mathbf{X}=(\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}=\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{pmatrix}=\begin{pmatrix} 2 & 4 \\ -1 & -2 \\ 3 & 3 \end{pmatrix}.$$

3.7.3. In each case, the given information implies that $rank\,(\mathbf{A})<n$—see the solution for Exercise 2.1.3.

3.7.4. (a) If $\mathbf{D}$ is diagonal, then $\mathbf{D}^{-1}$ exists if and only if each $d_{ii}\neq 0$, in which case

$$\begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}^{-1}=\begin{pmatrix} 1/d_{11} & 0 & \cdots & 0 \\ 0 & 1/d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_{nn} \end{pmatrix}.$$

(b) If $\mathbf{T}$ is triangular, then $\mathbf{T}^{-1}$ exists if and only if each $t_{ii}\neq 0$. If $\mathbf{T}$ is upper (lower) triangular, then $\mathbf{T}^{-1}$ is also upper (lower) triangular with $[\mathbf{T}^{-1}]_{ii}=1/t_{ii}$.

3.7.5. $\left(\mathbf{A}^{-1}\right)^T=\left(\mathbf{A}^T\right)^{-1}=\mathbf{A}^{-1}$.

3.7.6. Start with $\mathbf{A}(\mathbf{I}-\mathbf{A})=(\mathbf{I}-\mathbf{A})\mathbf{A}$ and apply $(\mathbf{I}-\mathbf{A})^{-1}$ to both sides, first on one side and then on the other.

3.7.7. Use the result of Example 3.6.5 that says that $trace\,(\mathbf{AB})=trace\,(\mathbf{BA})$ to write

$$m=trace\,(\mathbf{I}_m)=trace\,(\mathbf{AB})=trace\,(\mathbf{BA})=trace\,(\mathbf{I}_n)=n.$$

3.7.8. Use the reverse order law for inversion to write

$$\left[\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B}\right]^{-\mathbf{1}}=\mathbf{B}^{-1}(\mathbf{A}+\mathbf{B})\mathbf{A}^{-1}=\mathbf{B}^{-1}+\mathbf{A}^{-1}$$

and

$$\left[\mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}\right]^{-\mathbf{1}}=\mathbf{A}^{-1}(\mathbf{A}+\mathbf{B})\mathbf{B}^{-1}=\mathbf{B}^{-1}+\mathbf{A}^{-1}.$$

3.7.9. (a) $(\mathbf{I}-\mathbf{S})\mathbf{x}=\mathbf{0} \implies \mathbf{x}^T(\mathbf{I}-\mathbf{S})\mathbf{x}=0 \implies \mathbf{x}^T\mathbf{x}=\mathbf{x}^T\mathbf{S}\mathbf{x}$. Taking transposes on both sides yields $\mathbf{x}^T\mathbf{x}=-\mathbf{x}^T\mathbf{S}\mathbf{x}$, so that $\mathbf{x}^T\mathbf{x}=0$, and thus $\mathbf{x}=\mathbf{0}$

(recall Exercise 3.6.12). The conclusion follows from property (3.7.8).
(b) First notice that Exercise 3.7.6 implies that $\mathbf{A} = (\mathbf{I}+\mathbf{S})(\mathbf{I}-\mathbf{S})^{-1} = (\mathbf{I}-\mathbf{S})^{-1}(\mathbf{I}+\mathbf{S})$. By using the reverse order laws, transposing both sides yields exactly the same thing as inverting both sides.

3.7.10. Use block multiplication to verify that the product of the matrix with its inverse is the identity matrix.

3.7.11. Use block multiplication to verify that the product of the matrix with its inverse is the identity matrix.

3.7.12. Let $\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} \mathbf{D}^T & -\mathbf{B}^T \\ -\mathbf{C}^T & \mathbf{A}^T \end{pmatrix}$. The hypothesis implies that $\mathbf{MX} = \mathbf{I}$, and hence (from the discussion in Example 3.7.2) it must also be true that $\mathbf{XM} = \mathbf{I}$, from which the conclusion follows. **Note:** This problem appeared on a past Putnam Exam—a national mathematics competition for undergraduate students that is considered to be quite challenging. This means that you can be proud of yourself if you solved it before looking at this solution.

Solutions for exercises in section 3. 8

3.8.1. (a) $\mathbf{B}^{-1} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix}$

(b) Let $\mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{d}^T = (0 \quad 2 \quad 1)$ to obtain $\mathbf{C}^{-1} = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 3 & -1 \\ -1 & -4 & 2 \end{pmatrix}$

3.8.2. $\mathbf{A}_{*j}$ needs to be removed, and $\mathbf{b}$ needs to be inserted in its place. This is accomplished by writing $\mathbf{B} = \mathbf{A}+(\mathbf{b}-\mathbf{A}_{*j})\mathbf{e}_j^T$. Applying the Sherman–Morrison formula with $\mathbf{c} = \mathbf{b} - \mathbf{A}_{*j}$ and $\mathbf{d}^T = \mathbf{e}_j^T$ yields

$$\begin{aligned}\mathbf{B}^{-1} &= \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}(\mathbf{b}-\mathbf{A}_{*j})\mathbf{e}_j^T\mathbf{A}^{-1}}{1+\mathbf{e}_j^T\mathbf{A}^{-1}(\mathbf{b}-\mathbf{A}_{*j})} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{b}\mathbf{e}_j^T\mathbf{A}^{-1} - \mathbf{e}_j\mathbf{e}_j^T\mathbf{A}^{-1}}{1+\mathbf{e}_j^T\mathbf{A}^{-1}\mathbf{b} - \mathbf{e}_j^T\mathbf{e}_j} \\ &= \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{b}[\mathbf{A}^{-1}]_{j*} - \mathbf{e}_j[\mathbf{A}^{-1}]_{j*}}{[\mathbf{A}^{-1}]_{j*}\mathbf{b}} = \mathbf{A}^{-1} - \frac{\left(\mathbf{A}^{-1}\mathbf{b} - \mathbf{e}_j\right)[\mathbf{A}^{-1}]_{j*}}{[\mathbf{A}^{-1}]_{j*}\mathbf{b}}.\end{aligned}$$

3.8.3. Use the Sherman–Morrison formula to write

$$\begin{aligned}\mathbf{z} &= (\mathbf{A}+\mathbf{c}\mathbf{d}^T)^{-1}\mathbf{b} = \left(\mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{c}\mathbf{d}^T\mathbf{A}^{-1}}{1+\mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}}\right)\mathbf{b} = \mathbf{A}^{-1}\mathbf{b} - \frac{\mathbf{A}^{-1}\mathbf{c}\mathbf{d}^T\mathbf{A}^{-1}\mathbf{b}}{1+\mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}} \\ &= \mathbf{x} - \frac{\mathbf{y}\mathbf{d}^T\mathbf{x}}{1+\mathbf{d}^T\mathbf{y}}.\end{aligned}$$

3.8.4. (a) For a nonsingular matrix $\mathbf{A}$, the Sherman–Morrison formula guarantees that $\mathbf{A}+\alpha\mathbf{e}_i\mathbf{e}_j^T$ is also nonsingular when $1+\alpha\left[\mathbf{A}^{-1}\right]_{ji} \neq 0$, and this certainly will be true if α is sufficiently small.