MATH 4242 FALL 2016

Solution Guide for HOMEWORK 4

Meyer's Text:

3.9: 1, 2, 3, 5, 6, 9

3.10: 1, 2, 3, 8, 9, 10

Solutions for exercises in section 3. 9

3.9.1. (a) If $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_k$ is the sequence of elementary matrices that corresponds to the elementary row operations used in the reduction $[\mathbf{A}|\mathbf{I}] \longrightarrow [\mathbf{B}|\mathbf{P}]$, then

$$\begin{aligned} \mathbf{G}_k \cdots \mathbf{G}_2\mathbf{G}_1[\mathbf{A}|\mathbf{I}] = [\mathbf{B}|\mathbf{P}] &\implies [\mathbf{G}_k \cdots \mathbf{G}_2\mathbf{G}_1\mathbf{A} \mid \mathbf{G}_k \cdots \mathbf{G}_2\mathbf{G}_1\mathbf{I}] = [\mathbf{B}|\mathbf{P}] \\ &\implies \mathbf{G}_k \cdots \mathbf{G}_2\mathbf{G}_1\mathbf{A} = \mathbf{B} \quad \text{and} \quad \mathbf{G}_k \cdots \mathbf{G}_2\mathbf{G}_1 = \mathbf{P}. \end{aligned}$$

(b) Use the same argument given above, but apply it on the right-hand side.
(c) $[\mathbf{A}|\mathbf{I}] \xrightarrow{\text{Gauss–Jordan}} [\mathbf{E_A}|\mathbf{P}]$ yields

$$\left(\begin{array}{cccc|ccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 \\ 2 & 4 & 6 & 7 & 0 & 1 & 0 \\ 1 & 2 & 3 & 6 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|ccc} 1 & 2 & 3 & 0 & -7 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -5 & 2 & 1 \end{array}\right).$$

Thus $\mathbf{P} = \begin{pmatrix} -7 & 4 & 0 \\ 2 & -1 & 0 \\ -5 & 2 & 1 \end{pmatrix}$ is the product of the elementary matrices corresponding to the operations used in the reduction, and $\mathbf{PA} = \mathbf{E_A}$.
(d) You already have $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{E_A}$. Now find $\mathbf{Q}$ such that $\mathbf{E_A}\mathbf{Q} = \mathbf{N}_r$ by column reducing $\mathbf{E_A}$. Proceed using part (b) to accumulate $\mathbf{Q}$.

$$\left[\begin{array}{c} \mathbf{E_A} \\ \hline \mathbf{I}_4 \end{array}\right] \longrightarrow \left(\begin{array}{cccc} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

3.9.2. (a) Yes—because $rank(\mathbf{A}) = rank(\mathbf{B})$. (b) Yes—because $\mathbf{E_A} = \mathbf{E_B}$. (c) No—because $\mathbf{E}_{\mathbf{A}^T} \neq \mathbf{E}_{\mathbf{B}^T}$.

3.9.3. The positions of the basic columns in $\mathbf{A}$ correspond to those in $\mathbf{E_A}$. Because $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B} \iff \mathbf{E_A} = \mathbf{E_B}$, it follows that the basic columns in $\mathbf{A}$ and $\mathbf{B}$ must be in the same positions.

3.9.4. An elementary interchange matrix (a Type I matrix) has the form $\mathbf{E} = \mathbf{I} - \mathbf{uu}^T$, where $\mathbf{u} = \mathbf{e}_i - \mathbf{e}_j$, and it follows from (3.9.1) that $\mathbf{E} = \mathbf{E}^T = \mathbf{E}^{-1}$. If $\mathbf{P} = \mathbf{E}_1\mathbf{E}_2 \cdots \mathbf{E}_k$ is a product of elementary interchange matrices, then the reverse order laws yield

$$\begin{aligned} \mathbf{P}^{-1} &= (\mathbf{E}_1\mathbf{E}_2 \cdots \mathbf{E}_k)^{-1} = \mathbf{E}_k^{-1} \cdots \mathbf{E}_2^{-1}\mathbf{E}_1^{-1} \\ &= \mathbf{E}_k^T \cdots \mathbf{E}_2^T\mathbf{E}_1^T = (\mathbf{E}_1\mathbf{E}_2 \cdots \mathbf{E}_k)^T = \mathbf{P}^T. \end{aligned}$$

3.9.5. They are all true! $\mathbf{A} \sim \mathbf{I} \sim \mathbf{A}^{-1}$ because $rank\,(\mathbf{A}) = n = rank\,(\mathbf{A}^{-1})$, $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{A}^{-1}$ because $\mathbf{PA} = \mathbf{A}^{-1}$ with $\mathbf{P} = (\mathbf{A}^{-1})^2 = \mathbf{A}^{-2}$, and $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{A}^{-1}$ because $\mathbf{AQ} = \mathbf{A}^{-1}$ with $\mathbf{Q} = \mathbf{A}^{-2}$. The fact that $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{I}$ and $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{I}$ follows since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

3.9.6. (a), (c), (d), and (e) are true.

3.9.7. Rows i and j can be interchanged with the following sequence of Type II and Type III operations—this is Exercise 1.2.12 on p. 14.

$$\begin{array}{ll} R_j \leftarrow R_j + R_i & \text{(replace row } j \text{ by the sum of row } j \text{ and } i) \\ R_i \leftarrow R_i - R_j & \text{(replace row } i \text{ by the difference of row } i \text{ and } j) \\ R_j \leftarrow R_j + R_i & \text{(replace row } j \text{ by the sum of row } j \text{ and } i) \\ R_i \leftarrow -R_i & \text{(replace row } i \text{ by its negative)} \end{array}$$

Translating these to elementary matrices (remembering to build from the right to the left) produces

$$(\mathbf{I} - 2\mathbf{e}_i\mathbf{e}_i^T)(\mathbf{I} + \mathbf{e}_j\mathbf{e}_i^T)(\mathbf{I} - \mathbf{e}_i\mathbf{e}_j^T)(\mathbf{I} + \mathbf{e}_j\mathbf{e}_i^T) = \mathbf{I} - \mathbf{u}\mathbf{u}^T, \quad \text{where} \quad \mathbf{u} = \mathbf{e}_i - \mathbf{e}_j.$$

3.9.8. Let $\mathbf{B}_{m\times r} = [\mathbf{A}_{*b_1}\mathbf{A}_{*b_2}\cdots\mathbf{A}_{*b_r}]$ contain the basic columns of $\mathbf{A}$, and let $\mathbf{C}_{r\times n}$ contain the nonzero rows of $\mathbf{E_A}$. If $\mathbf{A}_{*k}$ is basic—say $\mathbf{A}_{*k} = \mathbf{A}_{*b_j}$—then $\mathbf{C}_{*k} = \mathbf{e}_j$, and

$$(\mathbf{BC})_{*k} = \mathbf{BC}_{*k} = \mathbf{Be}_j = \mathbf{B}_{*j} = \mathbf{A}_{*b_j} = \mathbf{A}_{*k}.$$

If $\mathbf{A}_{*k}$ is nonbasic, then $\mathbf{C}_{*k}$ is nonbasic and has the form

$$\mathbf{C}_{*k} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_j \\ \vdots \\ 0 \end{pmatrix} = \mu_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \mu_j \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2 + \cdots + \mu_j\mathbf{e}_j,$$

where the $\mathbf{e}_i$'s are the basic columns to the left of $\mathbf{C}_{*k}$. Because $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{E_A}$, the relationships that exist among the columns of $\mathbf{A}$ are exactly the same as the relationships that exist among the columns of $\mathbf{E_A}$. In particular,

$$\mathbf{A}_{*k} = \mu_1\mathbf{A}_{*b_1} + \mu_2\mathbf{A}_{*b_2} + \cdots + \mu_j\mathbf{A}_{*b_j},$$

where the $\mathbf{A}_{*b_i}$'s are the basic columns to the left of $\mathbf{A}_{*k}$. Therefore,

$$\begin{aligned} (\mathbf{BC})_{*k} = \mathbf{BC}_{*k} &= \mathbf{B}\left(\mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2 + \cdots + \mu_j\mathbf{e}_j\right) \\ &= \mu_1\mathbf{B}_{*1} + \mu_2\mathbf{B}_{*2} + \cdots + \mu_j\mathbf{B}_{*j} \\ &= \mu_1\mathbf{A}_{*b_1} + \mu_2\mathbf{A}_{*b_2} + \cdots + \mu_j\mathbf{A}_{*b_j} \\ &= \mathbf{A}_{*k}. \end{aligned}$$

3.9.9. If $\mathbf{A} = \mathbf{u}\mathbf{v}^T$, where $\mathbf{u}_{m\times 1}$ and $\mathbf{v}_{n\times 1}$ are nonzero columns, then

$$\mathbf{u} \overset{row}{\sim} \mathbf{e}_1 \quad \text{and} \quad \mathbf{v}^T \overset{col}{\sim} \mathbf{e}_1^T \implies \mathbf{A} = \mathbf{u}\mathbf{v}^T \sim \mathbf{e}_1\mathbf{e}_1^T = \mathbf{N}_1 \implies rank(\mathbf{A}) = 1.$$

Conversely, if $rank(\mathbf{A}) = 1$, then the existence of $\mathbf{u}$ and $\mathbf{v}$ follows from Exercise 3.9.8. If you do not wish to rely on Exercise 3.9.8, write $\mathbf{PAQ} = \mathbf{N}_1 = \mathbf{e}_1\mathbf{e}_1^T$, where $\mathbf{e}_1$ is $m \times 1$ and $\mathbf{e}_1^T$ is $1 \times n$ so that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{e}_1\mathbf{e}_1^T\mathbf{Q}^{-1} = \left(\mathbf{P}^{-1}\right)_{*1}\left(\mathbf{Q}^{-1}\right)_{1*} = \mathbf{u}\mathbf{v}^T.$$

3.9.10. Use Exercise 3.9.9 and write

$$\mathbf{A} = \mathbf{u}\mathbf{v}^T \implies \mathbf{A}^2 = \left(\mathbf{u}\mathbf{v}^T\right)\left(\mathbf{u}\mathbf{v}^T\right) = \mathbf{u}\left(\mathbf{v}^T\mathbf{u}\right)\mathbf{v}^T = \tau\mathbf{u}\mathbf{v}^T = \tau\mathbf{A},$$

where $\tau = \mathbf{v}^T\mathbf{u}$. Recall from Example 3.6.5 that $trace(\mathbf{AB}) = trace(\mathbf{BA})$, and write

$$\tau = trace(\tau) = trace\left(\mathbf{v}^T\mathbf{u}\right) = trace\left(\mathbf{u}\mathbf{v}^T\right) = trace(\mathbf{A}).$$

Solutions for exercises in section 3. 10

3.10.1. (a) $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ and $\mathbf{U} = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}$ (b) $\mathbf{x}_1 = \begin{pmatrix} 110 \\ -36 \\ 8 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 112 \\ -39 \\ 10 \end{pmatrix}$

(c) $\mathbf{A}^{-1} = \frac{1}{6}\begin{pmatrix} 124 & -40 & 14 \\ -42 & 15 & -6 \\ 10 & -4 & 2 \end{pmatrix}$

3.10.2. (a) The second pivot is zero. (b) $\mathbf{P}$ is the permutation matrix associated with the permutation $\mathbf{p} = (2 \;\; 4 \;\; 1 \;\; 3)$. $\mathbf{P}$ is constructed by permuting the rows of $\mathbf{I}$ in this manner.

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/3 & 0 & 1 & 0 \\ 2/3 & -1/2 & 1/2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{U} = \begin{pmatrix} 3 & 6 & -12 & 3 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 8 & 16 \\ 0 & 0 & 0 & -5 \end{pmatrix}$$

(c) $\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

3.10.3. $\xi = 0, \ \pm\sqrt{2}, \ \pm\sqrt{3}$

3.10.4. $\mathbf{A}$ possesses an LU factorization if and only if all leading principal submatrices are nonsingular. The argument associated with equation (3.10.13) proves that

$$\begin{pmatrix} \mathbf{L}_k & \mathbf{0} \\ \mathbf{c}^T\mathbf{U}_k^{-1} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{U}_k & \mathbf{L}_k^{-1}\mathbf{b} \\ \mathbf{0} & a_{k+1,k+1} - \mathbf{c}^T\mathbf{A}_k^{-1}\mathbf{b} \end{pmatrix} = \mathbf{L}_{k+1}\mathbf{U}_{k+1}$$

is the LU factorization for $\mathbf{A}_{k+1}$. The desired conclusion follows from the fact that the $k+1^{th}$ pivot is the $(k+1, k+1)$-entry in $\mathbf{U}_{k+1}$. This pivot must be nonzero because $\mathbf{U}_{k+1}$ is nonsingular.

3.10.5. If $\mathbf{L}$ and $\mathbf{U}$ are both triangular with 1's on the diagonal, then $\mathbf{L}^{-1}$ and $\mathbf{U}^{-1}$ contain only integer entries, and consequently $\mathbf{A}^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1}$ is an integer matrix.

3.10.6. (b) $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{pmatrix}$ and $\mathbf{U} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 1/4 \end{pmatrix}$

3.10.7. Observe how the LU factors evolve from Gaussian elimination. Following the procedure described in Example 3.10.1 where multipliers ℓ_{ij} are stored in the positions they annihilate (i.e., in the (i,j)-position), and where $\star$'s are put in positions that can be nonzero, the reduction of a 5×5 band matrix with bandwidth $w = 2$ proceeds as shown below.

$$\begin{pmatrix} \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & 0 \\ \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \end{pmatrix} \longrightarrow \begin{pmatrix} \star & \star & \star & 0 & 0 \\ l_{21} & \star & \star & \star & 0 \\ l_{31} & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \end{pmatrix} \longrightarrow \begin{pmatrix} \star & \star & \star & 0 & 0 \\ l_{21} & \star & \star & \star & 0 \\ l_{31} & l_{32} & \star & \star & \star \\ 0 & l_{42} & \star & \star & \star \\ 0 & 0 & \star & \star & \star \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \star & \star & \star & 0 & 0 \\ l_{21} & \star & \star & \star & 0 \\ l_{31} & l_{32} & \star & \star & \star \\ 0 & l_{42} & l_{43} & \star & \star \\ 0 & 0 & l_{53} & \star & \star \end{pmatrix} \longrightarrow \begin{pmatrix} \star & \star & \star & 0 & 0 \\ l_{21} & \star & \star & \star & 0 \\ l_{31} & l_{32} & \star & \star & \star \\ 0 & l_{42} & l_{43} & \star & \star \\ 0 & 0 & l_{53} & l_{54} & \star \end{pmatrix}$$

Thus $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 & 0 \\ 0 & l_{42} & l_{43} & 1 & 0 \\ 0 & 0 & l_{53} & l_{54} & 1 \end{pmatrix}$ and $\mathbf{U} = \begin{pmatrix} \star & \star & \star & 0 & 0 \\ 0 & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \star & \star \\ 0 & 0 & 0 & 0 & \star \end{pmatrix}$.

3.10.8. (a) $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

3.10.9. (a) $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, and $\mathbf{U} = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

(b) Use the same argument given for the uniqueness of the LU factorization with minor modifications.

(c) $\mathbf{A} = \mathbf{A}^T \implies \mathbf{LDU} = \mathbf{U}^T\mathbf{D}^T\mathbf{L}^T = \mathbf{U}^T\mathbf{D}\mathbf{L}^T$. These are each LDU factorizations for $\mathbf{A}$, and consequently the uniqueness of the LDU factorization means that $\mathbf{U} = \mathbf{L}^T$.

3.10.10. $\mathbf{A}$ is symmetric with pivots 1, 4, 9. The Cholesky factor is $\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{pmatrix}$.