Solutions for Chapter 4

Solutions for exercises in section 4.1

4.1.1. Only (b) and (d) are subspaces.
4.1.2. (a), (b), (f), (g), and (i) are subspaces.
4.1.3. All of $\mathbb{R}^3$.
4.1.4. If $v \in V$ is a nonzero vector in a space $V$, then all scalar multiples $\alpha v$ must also be in $V$.
4.1.5. (a) A line. (b) The $(x,y)$-plane. (c) $\mathbb{R}^3$
4.1.6. Only (c) and (e) span $\mathbb{R}^3$. To see that (d) does not span $\mathbb{R}^3$, ask whether or not every vector $(x,y,z) \in \mathbb{R}^3$ can be written as a linear combination of the vectors in (d). It’s convenient to think in terms columns, so rephrase the question by asking if every $b = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can be written as a linear combination of $\begin{Bmatrix} v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} \end{Bmatrix}$. That is, for each $b \in \mathbb{R}^3$, are there scalars $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = b$ or, equivalently, is

$$
\begin{pmatrix}
1 & 2 & 4 \\
2 & 0 & 4 \\
1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix}
= 
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
$$

consistent for all $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$?

This is a system of the form $Ax = b$, and it is consistent for all $b$ if and only if $\text{rank } ([A|b]) = \text{rank } (A)$ for all $b$. Since

$$
\begin{pmatrix}
1 & 2 & 4 \\
2 & 0 & 4 \\
1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 4 \\
0 & -4 & -4 \\
0 & -3 & -3
\end{pmatrix}
\begin{pmatrix}
x \\
y - 2x \\
z - x
\end{pmatrix}
$$

$$
\rightarrow
\begin{pmatrix}
1 & 2 & 4 \\
0 & -4 & -4 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y - 2x \\
(x/2) - (3y/4) + z
\end{pmatrix}
$$

it's clear that there exist $b$'s (e.g., $b = (1,0,0)^T$) for which $Ax = b$ is not consistent, and hence not all $b$'s are a combination of the $v_i$'s. Therefore, the $v_i$'s don't span $\mathbb{R}^3$.

4.1.7. This follows from (4.1.2).
4.1.8. (a) \( u, v \in \mathcal{X} \cap \mathcal{Y} \implies u, v \in \mathcal{X} \) and \( u, v \in \mathcal{Y} \). Because \( \mathcal{X} \) and \( \mathcal{Y} \) are closed with respect to addition, it follows that \( u + v \in \mathcal{X} \) and \( u + v \in \mathcal{Y} \), and therefore \( u + v \in \mathcal{X} \cap \mathcal{Y} \). Because \( \mathcal{X} \) and \( \mathcal{Y} \) are both closed with respect to scalar multiplication, we have that \( \alpha u \in \mathcal{X} \) and \( \alpha u \in \mathcal{Y} \) for all \( \alpha \), and consequently \( \alpha u \in \mathcal{X} \cap \mathcal{Y} \) for all \( \alpha \).

(b) The union of two different lines through the origin in \( \mathbb{R}^2 \) is not a subspace.

4.1.9. (a) (A1) holds because \( x_1, x_2 \in A(S) \implies x_1 = As_1 \) and \( x_2 = As_2 \) for some \( s_1, s_2 \in S \implies x_1 + x_2 = A(s_1 + s_2) \). Since \( S \) is a subspace, it is closed under vector addition, so \( s_1 + s_2 \in S \). Therefore, \( x_1 + x_2 \) is the image of something in \( S \) — namely, \( s_1 + s_2 \) — and this means that \( x_1 + x_2 \in A(S) \). To see that (M1) holds, consider \( \alpha x \), where \( \alpha \) is an arbitrary scalar and \( x \in A(S) \). Now, \( x \in A(S) \implies x = As \) for some \( s \in S \implies \alpha x = \alpha As = A(\alpha s) \). Since \( S \) is a subspace, we are guaranteed that \( \alpha s \in S \), and therefore \( \alpha x \) is the image of something in \( S \). This is what it means to say \( \alpha x \in A(S) \).

(b) Prove equality by demonstrating that \( \text{span} \{As_1, As_2, \ldots, As_k\} \subseteq A(S) \) and \( A(S) \subseteq \text{span} \{As_1, As_2, \ldots, As_k\} \). To show \( \text{span} \{As_1, As_2, \ldots, As_k\} \subseteq A(S) \), write

\[
x \in \text{span} \{As_1, As_2, \ldots, As_k\} \implies x = \sum_{i=1}^{k} \alpha_i(As_i) = A \left( \sum_{i=1}^{k} \alpha_i s_i \right) \in A(S).
\]

Inclusion in the reverse direction is established by saying

\[
x \in A(S) \implies x = As \text{ for some } s \in S \implies s = \sum_{i=1}^{k} \beta_i s_i
\]

\[
\implies x = A \left( \sum_{i=1}^{k} \beta_i s_i \right) = \sum_{i=1}^{k} \beta_i(As_i) \in \text{span} \{As_1, As_2, \ldots, As_k\}.
\]

4.1.10. (a) Yes, all of the defining properties are satisfied.

(b) Yes, this is essentially \( \mathbb{R}^2 \).

(c) No, it is not closed with respect to scalar multiplication.

4.1.11. If \( \text{span} (\mathcal{M}) = \text{span} (\mathcal{N}) \), then every vector in \( \mathcal{N} \) must be a linear combination of vectors from \( \mathcal{M} \). In particular, \( v \) must be a linear combination of the \( m_i \)'s, and hence \( v \in \text{span} (\mathcal{M}) \). To prove the converse, first notice that \( \text{span} (\mathcal{M}) \subseteq \text{span} (\mathcal{N}) \). The desired conclusion will follow if it can be demonstrated that \( \text{span} (\mathcal{M}) \supseteq \text{span} (\mathcal{N}) \). The hypothesis that \( v \in \text{span} (\mathcal{M}) \) guarantees that \( v = \sum_{i=1}^{r} \beta_i m_i \). If \( z \in \text{span} (\mathcal{N}) \), then

\[
z = \sum_{i=1}^{r} \alpha_i m_i + \alpha_{r+1} v = \sum_{i=1}^{r} \alpha_i m_i + \alpha_{r+1} \sum_{i=1}^{r} \beta_i m_i
\]

\[
= \sum_{i=1}^{r} \left( \alpha_i + \alpha_{r+1} \beta_i \right) m_i,
\]
which shows \( z \in \text{span} (\mathcal{M}) \), and therefore \( \text{span} (\mathcal{M}) \supseteq \text{span} (\mathcal{N}) \).

4.1.12. To show \( \text{span} (\mathcal{S}) \subseteq \mathcal{M} \), observe that \( x \in \text{span} (\mathcal{S}) \implies x = \sum \alpha_i v_i \). If \( \mathcal{V} \) is any subspace containing \( \mathcal{S} \), then \( \sum \alpha_i v_i \in \mathcal{V} \) because \( \mathcal{V} \) is closed under addition and scalar multiplication, and therefore \( x \in \mathcal{M} \). The fact that \( \mathcal{M} \subseteq \text{span} (\mathcal{S}) \) follows because if \( x \in \mathcal{M} \), then \( x \in \text{span} (\mathcal{S}) \) because \( \text{span} (\mathcal{S}) \) is one particular subspace that contains \( \mathcal{S} \).

Solutions for exercises in section 4.2

4.2.1. \( R(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\} \), \( N(A^T) = \text{span} \left\{ \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \right\} \), \( N(A) = \text{span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \), \( R(A^T) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\} \).

4.2.2. (a) This is simply a restatement of equation (4.2.3).
(b) \( Ax = b \) has a unique solution if and only if \( \text{rank} (A) = n \) (i.e., there are no free variables—see §2.5), and (4.2.10) says \( \text{rank} (A) = n \iff N(A) = \{0\} \).

4.2.3. (a) It is consistent because \( b \in R(A) \).
(b) It is nonunique because \( N(A) \neq \{0\} \)—see Exercise 4.2.2.

4.2.4. Yes, because \( \text{rank}[A|b] = \text{rank} (A) = 3 \implies \exists x \) such that \( Ax = b \) —i.e., \( Ax = b \) is consistent.

4.2.5. (a) If \( R(A) = \mathbb{R}^n \), then
\[ R(A) = R(I_n) \implies A^\text{col} \cong I_n \implies \text{rank} (A) = \text{rank} (I_n) = n. \]
(b) \( R(A) = R(A^T) = \mathbb{R}^n \) and \( N(A) = N(A^T) = \{0\} \).

4.2.6. \( E_A \neq E_B \) means that \( R(A^T) \neq R(B^T) \) and \( N(A) \neq N(B) \). However, \( E_{A^T} = E_{B^T} \) implies that \( R(A) = R(B) \) and \( N(A^T) = N(B^T) \).

4.2.7. Demonstrate that \( \text{rank} (A_{n \times n}) = n \) by using (4.2.10). If \( x \in N(A) \), then
\[ Ax = 0 \implies A_1 x = 0 \text{ and } A_2 x = 0 \implies x \in N(A_1) = R(A_2^T) \implies \exists y^T \text{ such that } x^T = y^T A_2 \implies x^T x = y^T A_2 x = 0 \implies \sum_i x_i^2 = 0 \implies x = 0. \]