Solutions for Chapter 4

Solutions for exercises in section 4. 1

4.1.1. Only (b) and (d) are subspaces.

4.1.2. (a), (b), (f), (g), and (i) are subspaces.

4.1.3. All of $\Re^3$.

4.1.4. If $\mathbf{v} \in \mathcal{V}$ is a nonzero vector in a space $\mathcal{V}$, then all scalar multiples $\alpha\mathbf{v}$ must also be in $\mathcal{V}$.

4.1.5. (a) A line. (b) The (x,y)-plane. (c) $\Re^3$

4.1.6. Only (c) and (e) span $\Re^3$. To see that (d) does not span $\Re^3$, ask whether or not every vector $(x, y, z) \in \Re^3$ can be written as a linear combination of the vectors in (d). It's convenient to think in terms columns, so rephrase the question by asking if every $\mathbf{b} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can be written as a linear combination of $\left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} \right\}$. That is, for each $\mathbf{b} \in \Re^3$, are there scalars $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{b}$ or, equivalently, is

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 4 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{consistent for all} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}?$$

This is a system of the form $\mathbf{Ax} = \mathbf{b}$, and it is consistent for all $\mathbf{b}$ if and only if $rank([\mathbf{A}|\mathbf{b}]) = rank(\mathbf{A})$ for all $\mathbf{b}$. Since

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & x \\ 2 & 0 & 4 & y \\ 1 & -1 & 1 & z \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & x \\ 0 & -4 & -4 & y - 2x \\ 0 & -3 & -3 & z - x \end{array}\right)$$
$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & x \\ 0 & -4 & -4 & y - 2x \\ 0 & 0 & 0 & (x/2) - (3y/4) + z \end{array}\right),$$

it's clear that there exist $\mathbf{b}$'s (e.g., $\mathbf{b} = (1, 0, 0)^T$) for which $\mathbf{Ax} = \mathbf{b}$ is not consistent, and hence not all $\mathbf{b}$'s are a combination of the $\mathbf{v}_i$'s. Therefore, the $\mathbf{v}_i$'s don't span $\Re^3$.

4.1.7. This follows from (4.1.2).

4.1.8. (a) $\mathbf{u}, \mathbf{v} \in \mathcal{X} \cap \mathcal{Y} \implies \mathbf{u}, \mathbf{v} \in \mathcal{X}$ and $\mathbf{u}, \mathbf{v} \in \mathcal{Y}$. Because $\mathcal{X}$ and $\mathcal{Y}$ are closed with respect to addition, it follows that $\mathbf{u}+\mathbf{v} \in \mathcal{X}$ and $\mathbf{u}+\mathbf{v} \in \mathcal{Y}$, and therefore $\mathbf{u}+\mathbf{v} \in \mathcal{X} \cap \mathcal{Y}$. Because $\mathcal{X}$ and $\mathcal{Y}$ are both closed with respect to scalar multiplication, we have that $\alpha\mathbf{u} \in \mathcal{X}$ and $\alpha\mathbf{u} \in \mathcal{Y}$ for all α, and consequently $\alpha\mathbf{u} \in \mathcal{X} \cap \mathcal{Y}$ for all α.

(b) The union of two different lines through the origin in $\Re^2$ is not a subspace.

4.1.9. (a) **(A1)** holds because $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{A}(\mathcal{S}) \implies \mathbf{x}_1 = \mathbf{A}\mathbf{s}_1$ and $\mathbf{x}_2 = \mathbf{A}\mathbf{s}_2$ for some $\mathbf{s}_1, \mathbf{s}_2 \in \mathcal{S} \implies \mathbf{x}_1 + \mathbf{x}_2 = \mathbf{A}(\mathbf{s}_1 + \mathbf{s}_2)$. Since $\mathcal{S}$ is a subspace, it is closed under vector addition, so $\mathbf{s}_1 + \mathbf{s}_2 \in \mathcal{S}$. Therefore, $\mathbf{x}_1 + \mathbf{x}_2$ is the image of something in $\mathcal{S}$—namely, $\mathbf{s}_1 + \mathbf{s}_2$—and this means that $\mathbf{x}_1 + \mathbf{x}_2 \in \mathbf{A}(\mathcal{S})$. To see that **(M1)** holds, consider $\alpha\mathbf{x}$, where α is an arbitrary scalar and $\mathbf{x} \in \mathbf{A}(\mathcal{S})$. Now, $\mathbf{x} \in \mathbf{A}(\mathcal{S}) \implies \mathbf{x} = \mathbf{A}\mathbf{s}$ for some $\mathbf{s} \in \mathcal{S} \implies \alpha\mathbf{x} = \alpha\mathbf{A}\mathbf{s} = \mathbf{A}(\alpha\mathbf{s})$. Since $\mathcal{S}$ is a subspace, we are guaranteed that $\alpha\mathbf{s} \in \mathcal{S}$, and therefore $\alpha\mathbf{x}$ is the image of something in $\mathcal{S}$. This is what it means to say $\alpha\mathbf{x} \in \mathbf{A}(\mathcal{S})$.

(b) Prove equality by demonstrating that $span\{\mathbf{A}\mathbf{s}_1, \mathbf{A}\mathbf{s}_2, \ldots, \mathbf{A}\mathbf{s}_k\} \subseteq \mathbf{A}(\mathcal{S})$ and $\mathbf{A}(\mathcal{S}) \subseteq span\{\mathbf{A}\mathbf{s}_1, \mathbf{A}\mathbf{s}_2, \ldots, \mathbf{A}\mathbf{s}_k\}$. To show $span\{\mathbf{A}\mathbf{s}_1, \mathbf{A}\mathbf{s}_2, \ldots, \mathbf{A}\mathbf{s}_k\} \subseteq \mathbf{A}(\mathcal{S})$, write

$$\mathbf{x} \in span\{\mathbf{A}\mathbf{s}_1, \mathbf{A}\mathbf{s}_2, \ldots, \mathbf{A}\mathbf{s}_k\} \implies \mathbf{x} = \sum_{i=1}^{k} \alpha_i(\mathbf{A}\mathbf{s}_i) = \mathbf{A}\left(\sum_{i=1}^{k} \alpha_i \mathbf{s}_i\right) \in \mathbf{A}(\mathcal{S}).$$

Inclusion in the reverse direction is established by saying

$$\mathbf{x} \in \mathbf{A}(\mathcal{S}) \implies \mathbf{x} = \mathbf{A}\mathbf{s} \text{ for some } \mathbf{s} \in \mathcal{S} \implies \mathbf{s} = \sum_{i=1}^{k} \beta_i \mathbf{s}_i$$

$$\implies \mathbf{x} = \mathbf{A}\left(\sum_{i=1}^{k} \beta_i \mathbf{s}_i\right) = \sum_{i=1}^{k} \beta_i(\mathbf{A}\mathbf{s}_i) \in span\{\mathbf{A}\mathbf{s}_1, \mathbf{A}\mathbf{s}_2, \ldots, \mathbf{A}\mathbf{s}_k\}.$$

4.1.10. (a) Yes, all of the defining properties are satisfied.

(b) Yes, this is essentially $\Re^2$.

(c) No, it is not closed with respect to scalar multiplication.

4.1.11. If $span(\mathcal{M}) = span(\mathcal{N})$, then every vector in $\mathcal{N}$ must be a linear combination of vectors from $\mathcal{M}$. In particular, $\mathbf{v}$ must be a linear combination of the $\mathbf{m}_i$'s, and hence $\mathbf{v} \in span(\mathcal{M})$. To prove the converse, first notice that $span(\mathcal{M}) \subseteq span(\mathcal{N})$. The desired conclusion will follow if it can be demonstrated that $span(\mathcal{M}) \supseteq span(\mathcal{N})$. The hypothesis that $\mathbf{v} \in span(\mathcal{M})$ guarantees that $\mathbf{v} = \sum_{i=1}^{r} \beta_i \mathbf{m}_i$. If $\mathbf{z} \in span(\mathcal{N})$, then

$$\begin{aligned}\mathbf{z} &= \sum_{i=1}^{r} \alpha_i \mathbf{m}_i + \alpha_{r+1}\mathbf{v} = \sum_{i=1}^{r} \alpha_i \mathbf{m}_i + \alpha_{r+1} \sum_{i=1}^{r} \beta_i \mathbf{m}_i \\ &= \sum_{i=1}^{r} \Big(\alpha_i + \alpha_{r+1}\beta_i\Big)\mathbf{m}_i,\end{aligned}$$

which shows $\mathbf{z} \in span(\mathcal{M})$, and therefore $span(\mathcal{M}) \supseteq span(\mathcal{N})$.

4.1.12. To show $span(\mathcal{S}) \subseteq \mathcal{M}$, observe that $\mathbf{x} \in span(\mathcal{S}) \implies \mathbf{x} = \sum_i \alpha_i \mathbf{v}_i$. If $\mathcal{V}$ is any subspace containing $\mathcal{S}$, then $\sum_i \alpha_i \mathbf{v}_i \in \mathcal{V}$ because $\mathcal{V}$ is closed under addition and scalar multiplication, and therefore $\mathbf{x} \in \mathcal{M}$. The fact that $\mathcal{M} \subseteq span(\mathcal{S})$ follows because if $\mathbf{x} \in \mathcal{M}$, then $\mathbf{x} \in span(\mathcal{S})$ because $span(\mathcal{S})$ is one particular subspace that contains $\mathcal{S}$.

Solutions for exercises in section 4. 2

4.2.1. $R(\mathbf{A}) = span\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\2\end{pmatrix}\right\}$, $N(\mathbf{A}^T) = span\left\{\begin{pmatrix}4\\1\\-2\end{pmatrix}\right\}$,

$$N(\mathbf{A}) = span\left\{\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\-3\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\-4\\0\\1\end{pmatrix}\right\},$$

$$R(\mathbf{A}^T) = span\left\{\begin{pmatrix}1\\2\\0\\-2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\3\\4\end{pmatrix}\right\}.$$

4.2.2. (a) This is simply a restatement of equation (4.2.3).

(b) $\mathbf{Ax} = \mathbf{b}$ has a unique solution if and only if $rank(\mathbf{A}) = n$ (i.e., there are no free variables—see §2.5), and (4.2.10) says $rank(\mathbf{A}) = n \iff N(\mathbf{A}) = \{\mathbf{0}\}$.

4.2.3. (a) It is consistent because $\mathbf{b} \in R(\mathbf{A})$.

(b) It is nonunique because $N(\mathbf{A}) \neq \{\mathbf{0}\}$—see Exercise 4.2.2.

4.2.4. Yes, because $rank[\mathbf{A}|\mathbf{b}] = rank(\mathbf{A}) = 3 \implies \exists\, \mathbf{x}$ such that $\mathbf{Ax} = \mathbf{b}$—i.e., $\mathbf{Ax} = \mathbf{b}$ is consistent.

4.2.5. (a) If $R(\mathbf{A}) = \Re^n$, then

$$R(\mathbf{A}) = R(\mathbf{I}_n) \implies \mathbf{A} \overset{col}{\sim} \mathbf{I}_n \implies rank(\mathbf{A}) = rank(\mathbf{I}_n) = n.$$

(b) $R(\mathbf{A}) = R(\mathbf{A}^T) = \Re^n$ and $N(\mathbf{A}) = N(\mathbf{A}^T) = \{\mathbf{0}\}$.

4.2.6. $\mathbf{E_A} \neq \mathbf{E_B}$ means that $R(\mathbf{A}^T) \neq R(\mathbf{B}^T)$ and $N(\mathbf{A}) \neq N(\mathbf{B})$. However, $\mathbf{E}_{\mathbf{A}^T} = \mathbf{E}_{\mathbf{B}^T}$ implies that $R(\mathbf{A}) = R(\mathbf{B})$ and $N(\mathbf{A}^T) = N(\mathbf{B}^T)$.

4.2.7. Demonstrate that $rank(\mathbf{A}_{n\times n}) = n$ by using (4.2.10). If $\mathbf{x} \in N(\mathbf{A})$, then

$$\begin{aligned}\mathbf{Ax} = \mathbf{0} &\implies \mathbf{A}_1\mathbf{x} = \mathbf{0} \quad \text{and} \quad \mathbf{A}_2\mathbf{x} = \mathbf{0}\\ &\implies \mathbf{x} \in N(\mathbf{A}_1) = R(\mathbf{A}_2^T) \implies \exists\, \mathbf{y}^T \text{ such that } \mathbf{x}^T = \mathbf{y}^T\mathbf{A}_2\\ &\implies \mathbf{x}^T\mathbf{x} = \mathbf{y}^T\mathbf{A}_2\mathbf{x} = \mathbf{0} \implies \sum_i x_i^2 = 0 \implies \mathbf{x} = \mathbf{0}.\end{aligned}$$