Math 4242: Linear Algebra
MidtermIII, December 14 2016

Directions:

PLEASE DO NOT OPEN EXAM UNTIL DIRECTED TO DO SO.
This is a closed book exam. No books. No notes. No crib sheets. No calculators.

You are allowed 50 minutes to complete this exam.

Please show all your work on the enclosed pages. You are not allowed any scratch paper of your own.

There are 7 questions. Including this title page, there are 11 pages (the last two of which are blank).

Please make sure all the pages are here before beginning your 50 minutes of work.

Scores:

(1) (4 pts) ____________

(2) (4 pts) ____________

(3) (5 pts) ____________

(4) (3 pts) ____________

(5) (3 pts) ____________

(6) (3 pts) ____________

(7) (5 pts) ____________

(1) (4 points) Consider the matrix

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix}.$$

Find an orthogonal matrix Q so that $Q^{-1}AQ$ is a diagonal matrix. Show all work,and also write down the resulting diagonal matrix.

(2) (4 pts) Consider the following three vectors $\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and $\vec{c} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$. Let P be the plane spanned by the vectors $\vec{a}, \vec{b}$.

Find the point $\vec{v_*}$ on the plane P which is closest to the vector $\vec{c}$. Show all your work.

(3) The purpose of this problem is to compute the Singular Value Decomposition (as described in the Olver/Shakiban text, for example) of the following matrix:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$$

That is, we wish to find matrices P, Q with orthonormal columns and a matrix Σ so that

$$A = P\Sigma Q^T$$

(a). (2 points) Find the matrix Q. Show all your work.

(b). (2 points) What are the singular values of the matrix A? What is the matrix Σ?

Singular Values = ____________.

Σ =

(c) (continued from previous page) (1 point) Find the matrix P which, along with your answers for Σ, Q above, insures that $A = P\Sigma Q^T$.

These next 3 pages have true or false questions. You should give brief (at most 5 sentences) reasons for your answers to each question. Each answer is worth 1/2 point, and you must write "True" or "False" on the line next to the problem number in order to receive this 1/2 point. Each "reason for answer" you give is worth 5/2 points.

If you say "true", you need to give a good explanation of why this is always true (i.e.: if you say "true", it's NOT enough to give a single example where it is true). If you say "false", you should give a counterexample.

(4) ____. (3 pts) True or False: Assume $A \in \mathrm{Mat}(n \times n)$ is real and suppose there is an orthonomal basis of R^n consisting of eigenvectors of A. Then we can always conclude that A is symmetric.

(Note: please do not answer, "This was proved in class." or "This is the Spectral Theorem". (Neither of those answers is valid.) The question here can be restated: is the **converse** of the Spectral Theorem true?)

(5) .____. (3 pts) True or False: If A is an orthogonal matrix and A is similar to the matrix B, then B is also orthogonal.

(6) .____. (3 pts) True or False: There is a 3×3 matrix A that satisfies

$$A^2 = -I$$

where here we have written I for the 3×3 identity matrix.

(7) (5 points) Suppose $P \in \mathrm{Mat}(n \times n)$ satisfies

$$P^2 = P.$$

Prove that,

$$\mathrm{rank}(P) = \mathrm{trace}(P).$$

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