Key

Math 4242: Linear Algebra
MidtermI : February 23, 2018

Directions:

PLEASE DO NOT OPEN EXAM UNTIL DIRECTED TO DO SO.
This is a closed book exam. No books. No notes. No crib sheets. No calculators.

You are allowed 50 minutes to complete this exam.

Please show all your work on the enclosed pages. You are not allowed any scratch paper of your own.

There are 7 questions. Including this title page, there are 11 pages (the last two of which are blank).

Please make sure all the pages are here before beginning your 50 minutes of work.

Scores:

(1) (5 pts) ________

(2) (3 pts) ________

(3) (5 pts) ________

(4) (5 pts) ________

(5) (5 pts) ________

(6) (5 pts) ________

(7) (5 pts) ________

Thanks to following Students for Solutions.
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(1) (5 pts) Consider the following linear system of equations,

$$x + \frac{1}{2}y + \frac{7}{2}z = \frac{5}{2}$$
$$2x - y + z = 3.$$

Using Gaussian elimination **with partial pivoting** on an augmented matrix, determine whether the system is consistent. If the system is consistent, find all solutions. If it is not consistent, explain why. Show all the steps in your work and be sure to use partial pivoting in your work (and to show this work on these exam pages). Partial Pivoting → switch row to highest magnitude top

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{7}{2} & \frac{5}{2} \\ 2 & -1 & 1 & 3 \end{array}\right) \xrightarrow{R2 \leftrightarrow R1} \left(\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 1 & 1/2 & 7/2 & 5/2 \end{array}\right) \xrightarrow{R2 - \frac{1}{2}R1}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & 1 & 6/2 & 2/2 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & 1 & 3 & 1 \end{array}\right)$$

The system is consistent.

Using back substitution,

$z = \text{free}$

$y + 3z = 1 \Rightarrow y = 1 - 3z$

$2x - y + z = 3$

$2x - (1 - 3z) + z = 3$

$2x - 1 + 4z = 3$

$x = 2 - 2z$

Solutions: the set $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} \,\middle|\, \lambda \in \mathbb{R} \right\}$

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Using Gaussian elimination **with partial pivoting** on an augmented matrix, determine whether the system is consistent. If the system is consistent, find all solutions. If it is not consistent, explain why. Show all the steps in your work and be sure to use partial pivoting in your work (and to show this work on these exam pages).

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{7}{2} & \frac{5}{2} \\ 2 & -1 & 1 & 3 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 1 & \frac{1}{2} & \frac{7}{2} & \frac{5}{2} \end{array}\right] \overset{-1/2 R_1 + R_2}{\sim} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & 1 & 3 & 1 \end{array}\right].$$

Consistent system, ∞ many solutions

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}}$$

$x_3 = x_3$

$x_2 = 1 - 3x_3$

$2x_1 = 3 + x_2 - x_3$

$x_1 = \dfrac{3 + (1 - 3x_3) - x_3}{2}$

$= 2 - 2x_3$

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(2) . (3pts) Suppose that

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

Write down a single Matrix B which, when we multiply A on the right by B, it adds 3 times the 2nd column of A to the 3rd column of A, and also switches the first column and the fourth column of A. In other words, find a matrix B so that

$$AB = \begin{pmatrix} 1 & 2 & 7 & 1 \\ 1 & 1 & 4 & 2 \\ 2 & 1 & 4 & 0 \end{pmatrix}.$$

You need not explain your answer, but show any work that you do on these test pages. (Note: Very little partial credit will be given on this problem.)

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}$$

3

(3) (5 pts) Use Gauss-Jordan elimination to find out whether or not the following matrix A is invertible (nonsingular). If it is invertible, write down the value of A^{-1}. If it is not invertible, explain briefly why not.

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

You should show all work.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \rho_2+(-1)\rho_1 \\ \rho_3+(-1)\rho_1 \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ \rho_3+\rho_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right)$$

A is not invertible. $A \in Mat(3\times3)$, yet $rank(A) = 2$. Therefore, A is not invertible.

5! Nice!

(4). (5 pts) True or False: If A and B are symmetric matrices, then their product AB is also symmetric.

A, B are symmetric $\rightarrow$ $A^T = A$, $B^T = B$

$$(AB)^T = B^T A^T = BA$$

we found that $(AB)^T = BA$

Only way that AB is symmetric is when $AB = BA$

$\therefore$ This is not always true.

Example.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is symetric $\quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is symetric $\quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \quad (AB)^T = \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix} \neq AB$

$\therefore$ AB is not symmetric #

Nice!

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$\boxed{\text{False}}$ #

(4). (5 pts) True or False: If A and B are symmetric matrices, then their product AB is also symmetric.

~~True to the matrix multiplication of the product, the element a_{ij} is the dot product of the ith row of A and the jth column of B. Its symmetric counterpart is a_{ji}, where this is the dot product of the jth row of A and the ith column of B.~~

$$\begin{array}{ccccc} A & & B & & AB \\ \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} & \cdot & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & = & \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \end{array}$$

False – A and B are both symmetric but the product AB is not

5 Nice!

(5). (5 pts) True or False: If A is an arbitrary $m \times n$ matrix, then the product AA^* is Hermitian. (Your answer also must carefully define what it means to be a Hermitian matirx.)

Hermitian: $A = A^* = \overline{A^T}$ complex conjugate transpose

Let $S = AA^*$

$S^* = (AA^*)^*$

$S^* = (A^*)^* A^*$

$S^* = AA^*$

$S = AA^*$

$S^* = AA^*$

$S = S^*$ so AA^* is Hermitian

True

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simple example

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A^* = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$AA^* = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$

hermitian

(5). (5 pts) True or False: If A is an arbitrary $m \times n$ matrix, then the product AA^* is Hermitian. (Your answer also must carefully define what it means to be a Hermitian matirx.)

means $A^* = A$

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$$AA^* \rightarrow (AA^*)^* = (A^*)^* A^* \quad ; \; (A^*)^* = A$$

$$= AA^*$$

Since $\underbrace{(AA^*)^*}_{\text{Same}} = \underbrace{AA^*}$

Nice!

We can conclude that AA^* is Hermitian #

Side note: $A \in \text{mat}(m \times n$

$A^* \in \text{mat}(n \times m)$

True

$AA^* \in \text{mat}(m \times m) \rightarrow$ A square matrix

$\therefore$ Hermitian property does exists #

SF

(6). (5 pts) True or False: If A is an $n \times n$ matrix such that $(I - A)$ is invertible (i.e. nonsingular), then we can always conclude that,

$$A(I - A)^{-1} = (I - A)^{-1}A.$$

(In other words, we can conclude that the matrices A and $(I - A)^{-1}$ must commute.)

True.

Since $(I-A)$ is invertible, $(I-A)^{-1}$ exists.

$A(I-A) = (I-A)A$ is true since $AI - A^2 = IA - A^2$

$\Leftrightarrow A - A^2 = A - A^2.$

multiply $(I-A)^{-1}$ on both side of $A(I-A)$, as well as $(I-A)A$

$(I-A)^{-1} A(I-A)(I-A)^{-1} = (I-A)^{-1}(I-A)A(I-A)^{-1}$

$\Leftrightarrow (I-A)AI = IA(I-A)^{-1}$

$\Leftrightarrow A(I-A)^{-1} = (I-A)^{-1}A.$

Nice! 5

(6). (5 pts) True or False: If A is an $n \times n$ matrix such that $(I - A)$ is invertible (i.e. nonsingular), then we can always conclude that,

$$A(I-A)^{-1} = (I-A)^{-1}A.$$

(In other words, we can conclude that the matrices A and $(I-A)^{-1}$ must commute.)

True!

Clearly, $A = A$. We can also say that $AI = IA$, and that $AI - A^2 = IA - A^2$. Using the distributive property carefully, $A(I-A) = (I-A)A$. Since $I-A$ is invertible, we can multiply $(I-A)^{-1}$ on the right on both sides to get

$$A(I-A)(I-A)^{-1} = (I-A)A(I-A)^{-1}!$$

Now let's multiply on the left on both sides by $(I-A)^{-1}$:

$$(I-A)^{-1}A(I-A)(I-A)^{-1} = (I-A)^{-1}(I-A)A(I-A)^{-1}$$

We have that $(I-A)(I-A)^{-1} = I$ by the definition of inverses, so:

$$(I-A)^{-1}AI = IA(I-A)^{-1}$$

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So, we are left with:

$$(I-A)^{-1}A = A(I-A)^{-1}$$

Nice!

as desired.

(7) (5 points) Suppose that $M = \{\vec{m}_1, \vec{m}_2, \ldots, \vec{m}_r\}$ and $N = \{\vec{m}_1, \vec{m}_2, \ldots, \vec{m}_r, \vec{v}\}$ are two sets of vectors from the same vector space. True or False: If we are told also that $\vec{v} \in \text{span}(M)$ then we can always conclude $\text{span}(M) = \text{span}(N)$.

$\boxed{\text{True}}$

If: $\vec{v} \in \text{span}(M)$, then $\vec{v} = \sum_{i=1}^{r} \alpha_i \vec{M}_i$ where α_i, is a scalar of the vector $\vec{m}_i$. Span(M) is all linear combinations of the vector.

$\text{Span}(M) = \sum_{i=1}^{r} (\beta_i) \vec{m}_i$. $\text{Span}(N) = \left(\sum_{i=1}^{r} \lambda_i \vec{M}_i\right) + T \cdot \vec{V}$

$= \sum_{i=1}^{r} \lambda_i M_i + T \sum_{i=1}^{r} (\alpha_i \vec{m}_i) = \sum_{i=1}^{r} (\lambda_i + T \cdot \alpha_i) \vec{m}_i$.

Consequently, span(N) is all linear combinations of the vectors of M. Since M and N share the same spanning vectors, they must share the same span. span(M) = span(N).

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Nice!