Quiz 2 Solution.

Math 4242; Spring 2018; Quiz 2: 20 minutes to complete.
Monday, February 5, 2018

(1) (4 points) Suppose that A is the coefficient matrix for a homogeneous system of 7 equations in 12 unknowns and suppose that A has at least one nonzero row.

a). Determine the fewest number of free variables (= non-basic variables) that are possible. (Explain your answer briefly, and circle your answer. No points assigned without a decent explanation.)

Solution. To determine the minimum number of free variables, we consider the row echelon form of A. The maximum number of non zero rows in the row echelon form of A is 7. We conclude that the maximum rank of A is 7. Then the minimum number of free variables is $12 - 7 = 5$. □

b). Determine the maximum number of free variables (= non-basic variable) that are possible. (Explain your answer briefly, and circle your answer. No points assigned without a decent explanation.)

Solution. Similarly to part a, A has one nonzero row so it's row echelon form has at least one nonzero row - hence it's rank is at least one, and it could be only one if all the other rows in the row echelon form are zero. In that case, the number of free variables would be $12 - 1 = 11$, so that is the maximum possible number of free variables. □

(2) (5 points) Find the general solution to the following system of linear equations. Use Gaussian Elimination to do this, and show all work, including your coefficient matrix in Row-Echelon form.

$$\begin{aligned} x_1 + 3x_2 + 2x_3 - x_4 &= 0 \\ 2x_1 + 6x_2 + x_3 + 4x_4 + 3x_5 &= 0 \\ -x_1 - 3x_2 - 3x_3 + 3x_4 + x_5 &= 0 \\ 3x_1 + 9x_2 + 8x_3 - 7x_4 + 2x_5 &= 0. \end{aligned}$$

Solution. We begin by writing the coefficient matrix for the system above as,

$$\begin{pmatrix} 1 & 3 & 2 & -1 & 0 \\ 2 & 6 & 1 & 4 & 3 \\ -1 & -3 & -3 & 3 & 1 \\ 3 & 9 & 8 & -7 & 2 \end{pmatrix}.$$

Next, we reduce the matrix above to row echelon form. Each row operation is written as $aR_i + R_j$, by which we mean a times row i added to row j. The result is then put in row j. The matrix is reduced as follows,

$$R_1 + R_3 \to \begin{pmatrix} 1 & 3 & 2 & -1 & 0 \\ 2 & 6 & 1 & 4 & 3 \\ 0 & 0 & -1 & 2 & 1 \\ 3 & 9 & 8 & -7 & 2 \end{pmatrix}$$

$$-2R_1 + R_2 \to \begin{pmatrix} 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & -3 & 3 & 3 \\ 0 & 0 & -1 & 2 & 1 \\ 3 & 9 & 8 & -7 & 2 \end{pmatrix}$$

$$-3R_3 + R_2 \to \begin{pmatrix} 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 3 & 9 & 8 & -7 & 2 \end{pmatrix}$$

$$-3R_1 + R_4 \to \begin{pmatrix} 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 2 & -4 & 2 \end{pmatrix}$$

$$2R_3 + R_4 \to \begin{pmatrix} 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \quad R.E.F \to \begin{pmatrix} 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From the row echelon form of our coefficient matrix, we note that the free variables are x_2 and x_4. We obtain expressions for x_1, x_3, and $x_5$using back substitution, but omit this step from the solution. The general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

□