Quiz 3 Solution
Math 4242

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February 13, 2018

1. (5 points) Compute the inverse of the following matrix using the Gauss-Jordan method. Show all your work clearly.

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

(solution on the next page.)

Solution. We will use the convention $aR_i + R_j$ to mean multiply Row i by a and add that to Row j. The result is then put in Row j.

$$\left(\begin{array}{ccc|ccc} 0 & 2 & 1 & 1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$-2R_3 + R_2 \rightarrow \left(\begin{array}{ccc|ccc} 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 4 & -7 & 0 & 1 & -2 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$-2R_1 + R_2 \rightarrow \left(\begin{array}{ccc|ccc} 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -9 & -2 & 1 & -2 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$-2R_1 + R_2 \rightarrow \left(\begin{array}{ccc|ccc} 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -9 & -2 & 1 & -2 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$-\tfrac{1}{9}R_2 \rightarrow \left(\begin{array}{ccc|ccc} 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/9 & -1/9 & 2/9 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$-R_2 + R_1 \rightarrow \left(\begin{array}{ccc|ccc} 0 & 2 & 0 & 7/9 & 1/9 & -2/9 \\ 0 & 0 & 1 & 2/9 & -1/9 & 2/9 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\tfrac{1}{2}R_1 \rightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 7/18 & 1/18 & -2/18 \\ 0 & 0 & 1 & 2/9 & -1/9 & 2/9 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$-R_1 + R_3 \rightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 7/18 & 1/18 & -2/18 \\ 0 & 0 & 1 & 2/9 & -1/9 & 2/9 \\ 1 & 0 & 4 & -7/18 & -1/18 & 20/18 \end{array}\right)$$

$$-4R_2 + R_3 \rightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 7/18 & 1/18 & -2/18 \\ 0 & 0 & 1 & 2/9 & -1/9 & 2/9 \\ 1 & 0 & 0 & -23/18 & 7/18 & 4/18 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -23/18 & 7/18 & 4/18 \\ 0 & 1 & 0 & 7/18 & 1/18 & -2/18 \\ 0 & 0 & 1 & 2/9 & -1/9 & 2/9 \end{array}\right)$$

2. (5 points) A square matrix $L = [l_{ij}]$ is said to be lower triangular if $l_{ij} = 0$ when $i < j$. Is it true that the product of two lower triangular matrices is lower triangular? Answer this clearly with a "Yes" or "No". If you answer 'yes', the give a complete explanation of why it is true. (If you say 'yes' you must show rigorously that the product $C = A \cdot B$ of two lower triangular matrices A, B is also lower triangular and you must show this using only the definition of 'lower triangular' and using the definition of matrix multiplication. Full credit will be given only if you provide an argument that doesn't rely exclusively on 'pictures' - but works with mathematical expressions and analyzes these mathematical expressions. Of course, a convincing argument using 'pictures' will get some partial credit.)

If you answer 'no' give an example to show why the statement is not true.

Solution. Yes.

Suppose A and B are $n \times n$ lower triangular matrices and that $i < j$. We show $(AB)_{ij} = 0$. We begin by expressing the product $(AB)_{ij}$ as,

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

We split the sum at $k = j - 1$ to reflect the lower triangular structure of A and B. This is realized as,

$$(AB)_{ij} = \sum_{k=1}^{j-1} A_{ik} B_{kj} + \sum_{k=j}^{n} A_{ik} B_{kj}.$$

Now, for $k < j$ $B_{kj} = 0$ since B is lower triangular. Then,

$$\sum_{k=1}^{j-1} A_{ik} B_{kj} = 0.$$

In the second sum, we have $k \geq j$. By assumption, $i < j$, so $i < j \leq k$. This implies $A_{ik} = 0$ since A is lower triangular. Then,

$$\sum_{k=j}^{n} A_{ik} B_{kj} = 0.$$

Therefore,

$$(AB)_{ij} = \sum_{k=1}^{j-1} A_{ik} B_{kj} + \sum_{k=j}^{n} A_{ik} B_{kj} = 0, \tag{1}$$

which is the definition of lower triangular. Therefore, if A and B are lower triangular, their product is also lower triangular as desired. □

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