(1) Let $X$ and $Y$ be two subspaces of a vector space $V$.

(a). (6 points) Prove that the intersection $X \cap Y$ is also a subspace of $V$.

Solution. For $X \cap Y$ to be a subspace of $V$, it must satisfy the following conditions,

1. For $u, v \in X \cap Y$, $u + v \in X \cap Y$
2. For $u \in X \cap Y$ and $\alpha \in \mathbb{R}$, $\alpha u \in X \cap Y$.

We begin by showing the first condition holds. Let $u, v \in X \cap Y$. Then $u, v \in X$ and $u, v \in Y$. Since both $X$ and $Y$ are subspaces of $V$, they are closed under addition. Therefore, $u + v \in X$ and $u + v \in Y$ which implies $u + v \in X \cap Y$, proving $X \cap Y$ is closed under addition.

We now show the second condition holds. Let $\alpha \in \mathbb{R}$. Then since $X$ and $Y$ are subspaces, they are closed under scalar multiplication. That is to say, $\alpha u \in X$ and $\alpha u \in Y$. Therefore, $\alpha u \in X \cap Y$, proving $X \cap Y$ is closed under scalar multiplication. Since $X \cap Y$ satisfies both conditions to be a subspace of $V$ and neither $X$ nor $Y$ are empty since they are subspaces, $X \cap Y$ is a subspace of $V$. $\square$

(b). (4 points) Show that the union $X \cup Y$ need not be a subspace.

Solution. To show that $X \cup Y$ is not a subspace of $V$, we provide a counter example. There are many different counter examples that work here. Consider two lines along the $x$ axis and $y$ axis and let $X$ and $Y$ be the subspaces created by these lines. The elements of $X$ and $Y$ are given by,

\[ \begin{pmatrix} a \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ b \end{pmatrix}, \]

respectively, where $a, b \in \mathbb{R}$. It is easy to verify that $X$ and $Y$ are subspaces of $\mathbb{R}^2$ and this step is omitted here. Let $u = (1, 0)^T$ and $v = (0, 1)^T$. Clearly, $u, v \in X \cup Y$ but $u + v \notin X \cup Y$ since

\[ u + v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \]

and the resulting vector is not an element of $X \cup Y$. Therefore, $X \cup Y$ is not closed under vector addition and thus, cannot be a subspace of $\mathbb{R}^2$. This shows that the union of two subspaces need not be a subspace. $\square$