Solutions

Math 4242; Fall 2016; Quiz 4: 10 minutes to complete.
Monday, October 17, 2016

(1) Let X and Y be two subspaces of a vector space V.

(a). (6 points) Prove that the intersection $X \cap Y$ is also a subspace of V.

Solution. For $X \cap Y$ to be a subspace of V, it must satisfy the following conditions,

1. For $u, v \in X \cap Y, u + v \in X \cap Y$
2. For $u \in X \cap Y$ and $\alpha \in \mathbb{R}, \alpha u \in X \cap Y$.

We begin by showing the first condition holds. Let $u, v \in X \cap Y$. Then $u, v \in X$ and $u, v \in Y$. Since both X and Y are subspaces of V, they are closed under addition. Therefore, $u + v \in X$ and $u + v \in Y$ which implies $u + v \in X \cap Y$, proving $X \cap Y$ is closed under addition.

We now show the second condition holds. Let $\alpha \in \mathbb{R}$. Then since X and Y are subspaces, they are closed under scalar multiplication. That is to say, $\alpha u \in X$ and $\alpha u \in Y$. Therefore, $\alpha u \in X \cap Y$, proving $X \cap Y$ is closed under scalar multiplication. Since $X \cap Y$ satisfies both conditions to be a subspace of V and neither X nor Y are empty since they are subspaces, $X \cap Y$ is a subspace of V. $\square$

(b). (4 points) Show that the union $X \cup Y$ need not be a subspace.

Solution. To show that $X \cup Y$ is not a subspace of V, we provide a counter example. There are many different counter examples that work here. Consider two lines along the x axis and y axis and let X and Y be the subspaces created by these lines. The elements of X and Y are given by,

$$\begin{pmatrix} a \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ b \end{pmatrix}, \tag{1}$$

respectively, where $a, b \in \mathbb{R}$. It is easy to verify that X and Y are subspaces of $\mathbb{R}^2$ and this step is omitted here. Let $u = (1, 0)^T$ and $v = (0, 1)^T$. Clearly, $u, v \in X \cup Y$ but $u + v \notin X \cup Y$ since

$$u + v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{2}$$

and the resulting vector is not an element of $X \cup Y$. Therefore, $X \cup Y$ is not closed under vector addition and thus, can not be a subspace of $\mathbb{R}^2$. This shows that the union of two subspaces need not be a subspace. $\square$