Solution Guide
Quiz 5

Math 4242; Spring 2018; Quiz 5: 20 minutes to complete.
Monday, March 5, 2018

(1) (10 points) Suppose that $A_{m\times n}$ is an $m \times n$ matrix which satisfies that

$$\sum_{j=1}^{n} a_{ij} = 0$$

for each $i = 1, 2, \ldots, m$. That is, each row sum is zero.

Explain why the columns of A are a linearly dependent set of vectors and explain also why we can conclude that $\mathrm{rank}(A) < n$.

$A = (a_{ij})$ and $\sum_{j=1}^{n} a_{ij} = 0$ for $i = 1, 2, \ldots, m$

This Implies that $A\vec{x}_0 = \vec{0}$

where $\vec{x}_0 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n$

i.e. $\vec{x}_0 \in \mathrm{Null}(A)$.

Hence $\dim(\mathrm{Null}(A)) \geq 1$ so there is at least one free (= non basic) column. So # of Pivots CANNOT equal the number of columns.

Hence # of Pivots $< n$

so $\mathrm{Rank}(A) < n$.

Also:

$A\vec{x}_0 = \vec{0}$, and also $A\vec{x}_0 =$ the sum of the column vectors.

This sum is a non-trivial linear combination of the column vectors

$\Rightarrow$ Columns linearly dependent.

(2) (10 points) Which of the following sets of functions are linearly independent. Show all your work.

(a). $\{\sin x, \cos x, x\sin x\}$

(b). $\{e^x, xe^x, x^2e^x\}$.

(c). $\{\sin^2 x, \cos^2 x, \cos 2x\}$.

(a) We look at the ~~Wronskian~~ Wronski Matrix.

$$W(x) = \begin{pmatrix} \sin x & \cos x & x\sin x \\ \cos x & -\sin x & \sin x + x\cos x \\ -\sin x & -\cos x & 2\cos x - x\sin x \end{pmatrix}$$

If there is one point where $W(x)$ is nonsingular, we can conclude the functions are linearly independent. Check at $x=0$ (Just a guess!!)

$$W(0) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

← Columns are linearly independent (can also put in REF to see 3 pivots!)

So: linearly independent!

(b) $$W(x) = \begin{pmatrix} e^x & xe^x & x^2e^x \\ e^x & e^x + xe^x & 2xe^x + x^2e^x \\ e^x & 2e^x + xe^x & 2e^x + 4xe^x + x^2e^x \end{pmatrix}$$

Try $x=0$

$$W(0) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix}$$

← Nonsingular!! Linearly independent

(c) We have by a trig identity that $\cos 2x = \cos^2 x - \sin^2 x$ so linearly dependent!