Solutions

Math 4242; Spring 2018 ; Quiz 6: 15 minutes to complete.
No books, no notes
Monday, March 26, 2018

(1) (15 points) Let T be the operator on $\mathbb{R}^2$ defined by

$$T(x, y) = (5x - 6y, 3x - 4y)$$

and let $\vec{v}_0$ be the fixed vector,

$$\vec{v}_0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Suppose we have a basis of $\mathbb{R}^2$ given by,

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

Determine $[T]_B$ and $[\vec{v}_0]_B$. You must show your work, and in addition to showing your work you must explain briefly why your answers are what you claim they are. (For example, if you simply write down an answer for $[\vec{v}_0]_B$ without writing down something about why this is the answer, you will get no credit for that answer.)

Solution. Let $\vec{b}_1$ and $\vec{b}_2$ be the basis vectors in B. To determine $[T]_B$, we evaluate T at both of the basis vectors of B as

$$\vec{b}_1 : \quad T(1, 1) = \begin{pmatrix} 5(1) - 6(1) \\ 3(1) - 4(1) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\vec{b}_2 : \quad T(2, 1) = \begin{pmatrix} 5(2) - 6(1) \\ 3(2) - 4(1) \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Next, we express $T(\vec{b}_1)$ and $T(\vec{b}_2)$ in terms of $\vec{b}_1$ and $\vec{b}_2$, as

$$T(\vec{b}_1) = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = -1\vec{b}_1 + 0\vec{b}_2$$

$$T(\vec{b}_2) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 0\vec{b}_1 + 2\vec{b}_2.$$

Finally, the columns of $[T]_B$ are given by the coefficients of $\vec{b}_1$ and $\vec{b}_2$ in the above equations. That is,

$$\boxed{[T]_B = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}} \qquad (1)$$

Since B is a basis, we must be able to express $\vec{v}_0$ in terms of that basis - that is, there must be some α and β so that

$$\vec{v}_0 = \alpha\vec{b}_1 + \beta\vec{b}_2.$$

In the coordinate system created by B, $\vec{v}_0$ is $[\vec{v}_0]_B$, and is given by,

$$[\vec{v}_0]_B = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

To determine α and β, we must solve the matrix equation,

(2) $$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix},$$

where the columns of the above matrix are the basis vectors of B. The solution to (2) is found using Gaussian elimination. The augmented coefficient matrix of the system is given by,

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 1 & 2 \end{array}\right).$$

We will use the convention $aR_i + R_j$ to mean a times row i added to row j. The result is then put in row j. The solution is then,

$$-1R_1 + R_2 \to \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -1 & -1 \end{array}\right)$$

$$2R_2 + R_1 \to \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & -1 \end{array}\right)$$

$$-1R_2 \to \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right).$$

Therefore, $\alpha = 1$ and $\beta = 1$. Then $[\vec{v}_0]_B$ is given by,

$$\boxed{[\vec{v}_0]_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

□