Math 4242; Spring 2018; Quiz 10: 15 minutes to complete.
Monday, April 16, 2018

Important: two pages to this quiz! 20 points total.

1). (5 points total)

a). (2 points) How many 2×2 matrices are both orthogonal and diagonal? Write all of them down, please.

Solution. A matrix B is orthogonal if $B^{-1} = B^T$. The 2×2 matrices which are both diagonal and orthogonal are:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Counting each matrix, it is evident that there are 4 2×2 matrices which are both diagonal and orthogonal. □

b). (3 points) How many $n \times n$ matrices are both orthogonal and diagonal? Give some indication of what they are.

Solution. Similarly to part a, the $n \times n$ matrices which are both orthogonal and diagonal have ± 1 on the diagonal. That is,

$$\begin{pmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & \pm 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pm 1 \end{pmatrix}.$$

Since there are n elements in the matrix and 2 possibilities for each element, there are 2^n $n \times n$ matrices which are both diagonal and orthogonal. □

2). (5 points) You may assume that $R^{n \times n}$ (the set of all $n \times n$ matrices with real entries) can be written as $R^{n \times n} = S \bigoplus K$ where S and K are the subspaces of $n \times n$ symmetric and skew-symmetric matrices, respectively.

Find the projection of A onto S along K where A is the matrix,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 7 & 8 & 9 \end{pmatrix}.$$

Solution. In general, this projection is given by,

$$P_A = \underbrace{\frac{A + A^T}{2}}_{\text{Symmetric}} + \underbrace{\frac{A - A^T}{2}}_{\text{Anti-Symmetric}}.$$

Since we are computing the projection of A onto S along K, we only need to compute the symmetric part of the above equation. Therefore, P_A is given by,

$$P_A = \frac{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 7 \\ 2 & 2 & 8 \\ 3 & 1 & 9 \end{pmatrix}}{2} = \frac{1}{2}\begin{pmatrix} 2 & 5 & 10 \\ 5 & 4 & 9 \\ 10 & 9 & 18 \end{pmatrix}$$

$$\rightarrow \boxed{P_A = \frac{1}{2}\begin{pmatrix} 2 & 5 & 10 \\ 5 & 4 & 9 \\ 10 & 9 & 18 \end{pmatrix}}$$

□

3). (10 points) Consider the following two vectors,

$$\vec{u} \; = \; \begin{pmatrix} 1 \\ -2 \\ 0 \\ 3 \end{pmatrix}, \qquad \vec{v} \; = \; \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix},$$

(a). (5 points) Determine the orthogonal projection of $\vec{u}$ onto span$\{\vec{v}\}$.

Solution. Let $\mathcal{M} = \mathrm{Span}(\vec{v})$. Then the projection of $\vec{u}$ onto $\mathcal{M}$ is given by,

$$P_{\mathcal{M}}\vec{u} = \frac{\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}}\vec{u} = \frac{\vec{v}^T\vec{u}}{\vec{v}^T\vec{v}}\vec{v} = \frac{\begin{pmatrix} 1 & 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 0 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 & 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}}\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \frac{0}{7}\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \vec{0}$$

$$\rightarrow \boxed{P_{\mathcal{M}}\vec{u} = \vec{0}}$$

□

(b). (5 points) Determine the orthogonal projection of $\vec{v}$ onto $\vec{u}^{\perp}$.

Solution. The projection of $\vec{v}$ onto $\vec{u}^{\perp}$ is given by,

$$P_{\vec{u}^{\perp}}\vec{v} = (\vec{v} - P_{\mathcal{M}}\vec{u}).$$

But $P_{\mathcal{M}}\vec{u} = \vec{0}$, so $P_{\vec{u}^{\perp}}\vec{v} = \vec{v}$. Therefore,

$$\boxed{P_{\vec{u}^{\perp}}\vec{v} = \vec{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}}$$

□