Key

Math 4242: Linear Algebra
MidtermI : October 12, 2016

Directions:

PLEASE DO NOT OPEN EXAM UNTIL DIRECTED TO DO SO.
This is a closed book exam. No books. No notes. No crib sheets. No calculators.

You are allowed 50 minutes to complete this exam.

Please show all your work on the enclosed pages. You are not allowed any scratch paper of your own.

There are 7 questions. Including this title page, there are 11 pages (the last two of which are blank).

Please make sure all the pages are here before beginning your 50 minutes of work.

Scores:

(1) (5 pts) ________

(2) (3 pts) ________

(3) (5 pts) ________

(4) (3 pts) ________

(5) (3 pts) ________

(6) (3 pts) ________

(7) (3 pts) ________

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for solutions

(1) (5 pts) Consider the following linear system of equations,

$$x_1 + 2x_2 - 3x_3 + x_4 = 1$$
$$-x_1 - x_2 + 4x_3 - x_4 = 6$$
$$-2x_1 - 4x_2 + 7x_3 - x_4 = 1$$

Using Gaussian elimination on an augmented matrix, determine whether the system is consistent. If the system is consistent, find all solutions. If it is not consistent, explain why. Show all your work.

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 1 \\ -1 & -1 & 4 & -1 & 6 \\ -2 & -4 & 7 & -1 & 1 \end{array}\right) \begin{array}{l} R_2^* = R_2 + R_1 \\ R_3^* = R_3 + 2R_1 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 1 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right) \begin{array}{l} R_1^* = R_1 - 2R_2 + 5R_3 \\ R_2^* = R_2 - R_3 \end{array}$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 6 & 2 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right)$$

CONSISTENT

↳ same # pivots as rows

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix} + s \begin{pmatrix} -6 \\ 1 \\ -1 \\ 1 \end{pmatrix}}$$

5

(2) . (3pts) Suppose that

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

Write down a single Matrix B which, when we multiply A on the left by B, it adds 2 times the 2nd row of A to the 3rd row of A, and also adds 4 times the first row of A to the second row of A. In other words, find a matrix B so that

$$BA = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 6 & 9 & 5 & 5 \\ 4 & 3 & 3 & 4 \end{pmatrix}$$

You need not explain your answer, but show any work that you do on these test pages.

$$\boxed{B = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}}$$

✓

Check:

$$BA = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 6 & 9 & 5 & 5 \\ 4 & 3 & 3 & 4 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

↑ 2 times 2nd row to 3rd

↑ 4 times first row to second

3

(3) (5 pts) Use Gauss-Jordan elimination to find A^{-1} if

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Show all work, though you do NOT need to explain what you are doing. Circle your answer.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-(R_1)+R_3]{-2(R_1)+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -3 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{-(R_2)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 3 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

5

$$\begin{bmatrix} -1 & 1 & 0 \\ 3 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(4) (3 pts) Determine the L-U decomposition of the following matrix using a method discussed in class and/or in the Meyer text. For full credit you must show all work.

$$A = \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix} \xrightarrow[R_2 - 1R_1]{} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

$$A = LU$$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \overset{\text{check}}{\underset{=}{\rightarrow}} \begin{pmatrix} 2 & 1 \\ 2 & 1+3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix}$$

3

TRUE OR FALSE

The following 3 pages have true or false questions. You must record your answers to these questions on THIS PAGE. You should give brief (at most 5 sentences) reasons for your answers on the following pages. Each answer is worth 1/2 point. Each "reason for answer" you give is worth 5/2 points.

If you say "true", you need to give a good explanation of why this is always true (i.e.: if you say "true", it's NOT enough to give a single example where it is true). If you say "false", you should give a counterexample.

WRITE "TRUE" OR "FALSE" ON EACH LINE.

(5) TRUE

(6) FALSE.

(7) TRUE.

(5). (3 pts) True or False: If A is nonsingular, then A and A^{-1} are row equivalent. ('Row equivalent' here means that one of these matrices can always be converted to the other using the three elementary row operations.)

True.

Proof: Suppose A is nonsingular. then A is invertable That is, $A \cdot A^{-1} = A^{-1} \cdot A = I$. And since A can be reduced to I by elementary row operations and suppose C be the elementary matrix corresponding to the elementary row operations used to reduce A to I. That is $CA = I$. Similarly, we have $BA^{-1} = I$. Hence $CA = BA^{-1}$. $\underline{B^{-1}C}\, A = A^{-1}$. Since C, B is both elementary matrices. they are invertible and the inverse is still elementary matrix. say D. So $DA = A^{-1}$ D is the elementary matrix corresponding to elementary row operations used to ~~reduce A~~ transform A to A^{-1}.

Very Nice!

3

A

(6). (3 pts) True or False: If A and B are two matrices that are row equivalent, then A^T is row equivalent to B^T.

$$A \overset{\text{row}}{\sim} B \implies A^T \overset{\text{row}}{\sim} B^T$$

false (should be col equiv)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}$$

Nice!

3/3

(6). (3 pts) True or False: If A and B are two matrices that are row equivalent, then A^T is row equivalent to B^T.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A and B are row equivalent

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A^T and B^T are not row equivalent.

Nice!

3/3

$1\times n \quad n\times n \quad n\times 1$

(7). (3 pts) True or False: Suppose S is a square, ~~antisymmetric~~ skew $n \times n$ matrix. Then for every $\vec{v} \in R^n$, we can be sure that,

$$\vec{v}^T S \vec{v} = 0.$$

True

thinking

$$v^T S v = 0 \qquad S^T = S$$

if $v^T S v = 0 \iff (v^T S v)^T = (0)^T = 0$

$\iff (v^T S^T v^{TT}) = 0. \iff v^T - S v = 0$

$\iff -v^T S v = 0$

thus $-v^T S v = 0 = v^T S v$ which is only true if $v^T S v = 0$ so yes, we are sure

actual proof

$x^T = (v^T S v)^T = v^T S^T v^{TT} = -v^T S v$

start

$x = v^T S v \quad x = x^T = (v^T S v)^T = (v^T S^T v^{TT}) = v^T(-S)v$

x is 1×1 scalar so $(x^T = x)$ $\quad = -v^T S v \quad$ so $x = v^T S v = -(v^T S v) = -x$

but the only value such that $x = -x$ is zero, so we are sure this is the case that $v^T S v = 0$ for all $v \in \mathbb{R}^n$.

Nice! 3/3