Math 4242: Linear Algebra
Midterm I: March 1, 2002

Directions:
PLEASE DO NOT OPEN EXAM UNTIL DIRECTED TO DO SO.
This is a closed book exam. No books. No notes. No crib sheets. No calculators.

You are allowed 60 minutes to complete this exam.

Please show all your work on the enclosed pages. You are not allowed any scratch paper of your own.

There are 11 questions. Including this title page, there are 11 pages.

Please make sure all the pages are here before beginning your 60 minutes of work.

Scores: (Each question is worth 5 points.)
(1) _________
(2) _________
(3) _________
(4) _________
(5) _________
(6) _________
(7) _________
(8) _________
(9) _________
(10) _________
(11) _________
(1) Suppose
\[
A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 1 & 2 \\ 2 & 2 & 0 & 2 \end{pmatrix}
\]

Write down the row reduced echelon form of \( A \), \text{RREF}(A).\ Show all work, but you need NOT explain what you are doing. Circle your answer.

\textbf{SOLUTION:} Add two times first row to the second row, and to the third row. Then multiply the resulting second row by \((-1)\), and then add two times that second row to the 3rd row. At this point you should have
\[
A \sim \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

Now clear out the second pivot column to give
\[
A \sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]
(2) Write down a basis for the nullspace of the following matrix. Show your work.

\[
\begin{pmatrix}
1 & 0 & 2 & 1 & 0 \\
0 & 1 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

**SOLUTION:** This matrix is already in RREF. The first, second, and fifth columns are pivot columns. The third and fourth columns are free. We get immediately then that the null space is given by

\[
\begin{pmatrix}
x_1 \\
x_2 \\
f_3 \\
f_4 \\
x_5
\end{pmatrix}
\begin{array}{c}
= -2f_3 - f_4; x_2 = -3f_3; x_5 = 0; f_3, f_4 \in \mathbb{R} \\
\end{array}
\]

In other words, the null space is

\[
\begin{pmatrix}
f_3 \\
1 \\
0 \\
0
\end{pmatrix}
+ \begin{pmatrix}
f_4 \\
0 \\
0 \\
0
\end{pmatrix}
\begin{array}{c}
\in \mathbb{R} \\
\end{array}
\]
(3) Suppose that

\[ A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \]

Write down a matrix \( B \) which, when we multiply \( A \) on the left by \( B \), it adds 2 times the 2nd row of \( A \) to the 3rd row of \( A \). In other words, find a matrix \( B \) so that

\[ BA = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 4 & 3 & 3 & 4 \end{pmatrix} \]

You need not explain your answer, but show any work.

**SOLUTION:** Using the notation we used in class, the matrix that we want is

\[ E_{32}(2) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \]
(4) Use Gauss-Jordan elimination to find $A^{-1}$ if

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{pmatrix}$$

Show all work, though you do NOT need to explain what you are doing. Circle your answer.

**SOLUTION:** This is a straightforward computation similar to many done on the homework. You first augment the matrix $A$ with the three by three identity matrix $I$, and then perform Gauss-Jordan elimination until the left side of this augmented matrix is the three by three identity matrix. At this point, you will find the matrix $A^{-1}$ on the right side of the augmented matrix. (And in class, we explained carefully why this matrix that ends up on the right side is indeed $A^{-1}$.)

$$A^{-1} = \begin{pmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{pmatrix}.$$ 

It is very strongly suggested that when finishing a problem like this, you go ahead and check that $AA^{-1} = I$. That is, check that in fact this is the inverse!
TRUE OR FALSE

The following pages have true or false questions. You must record your answers to these questions on THIS PAGE. You should give brief (at most 5 sentences) reasons for your answers on the following pages. Each answer is worth 1/2 point. Each “reason for answer” you give is worth 9/2 points.

WRITE “TRUE” OR “FALSE” ON EACH LINE.

(5) __________
(6) __________
(7) __________
(8) __________
(5). True or False: If $A$ is a $5 \times 6$ matrix (that is 5 rows and 6 columns) and the rank of $A$ is $5$, then there is at most one solution of $A\vec{x} = \vec{b}$.

SOLUTION: This is false. Note first that since rank($A$) = 5, the row rank of $A$ is also 5, and $A$ has “full row rank”. Therefore,

$$A\vec{x} = \vec{b}.$$ always has at least one solution. Pick one of these solutions and call it $\vec{x}_{\text{part}}$.

Since there are 6 columns in $A$ and the rank of $A$ is 5, one of these columns is a free column. Hence there are infinitely many vectors in the nullspace of $A$. Since the set

$$\vec{x}_{\text{part}} + \text{NULL}(A)$$
is infinite and describes all solutions of $A\vec{x} = \vec{b}$, there are infinitely many solutions.

(6). True or False: If a matrix $B$ is invertible, then the matrix $B^2 = BB$ is also invertible. (If you say “True”, then write down the inverse of $B^2$. If you say “False”, explain briefly why it is false, or give a counterexample.)

SOLUTION: This is true and immediate: we know that if $A, B$ are two invertible matrices, then $AB$ is also invertible, and $(AB)^{-1} = B^{-1}A^{-1}$. In this case, the two matrices are the same.

We have $B^2$ is invertible and $(BB)^{-1} = B^{-1}B^{-1} = (B^{-1})^2$
(7) True or False: It is possible to write down a matrix $A$ so that the column space is

$$\text{Col}(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

and the null space is

$$\text{Null}(A) = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

**SOLUTION:** This is FALSE: If such a matrix $A$ existed, we would have $A : \mathbb{R}^5 \rightarrow \mathbb{R}^4$. We know that the $\text{dim} \left( \text{NULL}(A) \right) + \text{dim} \left( \text{Row}(A) \right) = 5$. (This is true for any matrix with 5 columns!) Now, we compare this general fact with the information that we are given about the matrix $A$ we are trying to construct:

$$\text{dim} \left( \text{Row}(A) \right) = \text{dim} \left( \text{Col}(A) \right) = 2$$

and

$$\text{dim} \left( \text{Null}(A) \right) = 1.$$  

Hence if such a matrix $A$ existed, we would need to have

$$2 + 1 = 5$$

and of course, this is not possible! So no such matrix $A$ exists.
(8) True or False: If the $m \times m$ matrix $A$ is invertible, and the $m \times n$ matrix $B$ has full row rank, then the matrix $AB$ is invertible.

(If you say “true”, you do NOT have to write down the inverse - but give an explanation. If you say “False”, give an explanation.)

**SOLUTION:** False: The question is asking whether $AB$ will be invertible for any $A, B$ as described. If $n > m$, then the matrix $AB$ won’t be a square matrix, so it can’t possibly be invertible.
(9) Consider the plane in $\mathbb{R}^3$ defined by
\[ x + y + 3z = 0 \]

(a). Find a basis for the plane

**SOLUTION:** It suffices to find two linearly independent vectors in the plane, since the plane obviously has dimension 2. The two vectors
\[
\begin{pmatrix}
1 \\
-1 \\
0
\end{pmatrix}, \begin{pmatrix}
-3 \\
0 \\
1
\end{pmatrix}
\]
are an example of such a pair. (There are infinitely many pairs which are correct!)

(b). Find a basis for all vectors perpendicular to this plane.

**SOLUTION:** The plane is defined as all vectors in $\mathbb{R}^3$ which are perpendicular to the vector
\[
\begin{pmatrix}
1 \\
1 \\
3
\end{pmatrix}
\]
So, there the vector space of vectors perpendicular to the plane is spanned by that vector, and it therefore forms a basis.
(10) Assume that $A$ is an $m \times n$ matrix with rank $r$.
Suppose that there are vectors $\vec{b}$ so that the equation

$$A\vec{x} = \vec{b}$$

has no solution. Then which of the following statements must be true (CIRCLE TWO and give a brief explanation)

- $r \leq n$  $n \leq r$  $r < n$  $n < r$
- $r = m$  $m \leq r$  $r < m$  $m < r$
- $m \leq n$  $n \leq m$  $m < n$  $n < m$

**SOLUTION:** We have

$$A : \mathbb{R}^n \rightarrow \mathbb{R}^m.$$ 
It is always true that the rank of a matrix is less than or equal to the number of rows in the matrix. Therefore, $r \leq n$. (And it’s NOT NECESSARILY TRUE THAT $r < n$ - you might have $r = n$.)

The other thing which must be true is $r < m$: we are told that there are vectors $\vec{b}$ for which the system is inconsistent. Therefore, we must have that there are vectors $\vec{b} \in \mathbb{R}^m$ which are not in the column space of $A$. Therefore, the dimension of the column space must be less than $m$. Hence $r < m$.

(11) The equation

$$A^T\vec{y} = \vec{d}$$

is solvable when the right side $\vec{d}$ is in which of the four fundamental subspaces? Explain.

**SOLUTION:** The given equation means that $\vec{d}$ is a linear combination of the columns of $A^T$. But this is the same as being a linear combination of the rows of $A$, since the columns of $A^T$ are precisely the rows of $A$. Hence $\vec{d}$ is in the row space of $A$. 