PRACTICE WITH SET INCLUSION

For $n \in \mathbb{N}$, let $B_n = (-\frac{1}{n}, \frac{1}{n})$ and $A_n = (-\frac{1}{n}, 1 + \frac{1}{n})$. Then we can prove the following set equalities using set inclusion arguments:

$$\bigcap_{n=1}^{\infty} B_n = \{0\}$$

**Proof:** First, we wish to show that $\{0\} \subseteq \bigcap_{n=1}^{\infty} B_n$. To do so, we note that 0 is in the interval $(-\frac{1}{n}, \frac{1}{n})$ for all integers $n$. Since that means that $0 \in B_n$ for all $n$, clearly we have $0 \in \bigcap_{n=1}^{\infty} B_n$. So $\{0\} \subseteq \bigcap_{n=1}^{\infty} B_n$.

Next, we wish to show that $\bigcap_{n=1}^{\infty} B_n \subseteq \{0\}$. Consider some element $x \in \bigcap_{n=1}^{\infty} B_n$ and assume that $x$ is non-zero, so $|x| > 0$. Then there exists some integer $N$ such that $|x| > \frac{1}{N}$. But this means that $x$ is not in the interval $B_N = (-\frac{1}{N}, \frac{1}{N})$, so $x$ can’t be an element of the intersection $\bigcap_{n=1}^{\infty} B_n$. Because this contradicts our original assumption that $x$ was an element of the intersection, we conclude that 0 is the only possible element of the intersection. So we have $\bigcap_{n=1}^{\infty} B_n \subseteq \{0\}$.

Since we’ve shown both inclusions, we conclude that $\bigcap_{n=1}^{\infty} B_n = \{0\}$, as desired.

$$\bigcup_{n=1}^{\infty} B_n = (-1, 1)$$

**Proof:** First, we wish to show that $\bigcup_{n=1}^{\infty} B_n \subseteq (-1, 1)$. To do so, consider some element $x \in \bigcup_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n})$. Because $x$ is in the union, there exists at least one integer $N$ such that $x$ is in the interval $(-\frac{1}{N}, \frac{1}{N})$. Because $N \geq 1$, we know that $(-\frac{1}{N}, \frac{1}{N}) \subseteq (-1, 1)$. Therefore, $x$ is in the interval $(-1, 1)$ and we have the inclusion $\bigcup_{n=1}^{\infty} B_n \subseteq (-1, 1)$.
Next, we wish to show that \((-1, 1) \subseteq \bigcup_{n=1}^{\infty} B_n\). Suppose that \(x\) is an element of the interval \((-1, 1)\). Then \(x\) is in the interval \((-\frac{1}{n}, \frac{1}{n})\) where \(n = 1\), which is one of the intervals in our union. So clearly \((-1, 1) \subseteq \bigcup_{n=1}^{\infty} B_n\).

Since we’ve shown both inclusions, we conclude that \(\bigcup_{n=1}^{\infty} B_n = (-1, 1)\).

\[
\bigcap_{n=1}^{\infty} A_n = [0, 1]
\]

**Proof:** First, we’ll show that \(\bigcap_{n=1}^{\infty} A_n \subseteq [0, 1]\). Consider some element \(x\) of \(\bigcap_{n=1}^{\infty} A_n\). Suppose that \(x < 0\). Then there exists some integer \(N\) such that \(x < -\frac{1}{N} < 0\), so \(x\) is not in the interval \(A_N = (-\frac{1}{N}, 1 + \frac{1}{N})\) and therefore not in the intersection. Alternately, suppose that \(x > 1\). Then there exists some integer \(N'\) such that \(1 < 1 + \frac{1}{N'} < x\), so \(x\) is not in the interval \(A_{N'} = (-\frac{1}{N'}, 1 + \frac{1}{N'})\) and therefore not in the intersection. Since we’ve shown that the intersection cannot contain elements less than zero or greater than one, we have the inclusion \(\bigcap_{n=1}^{\infty} A_n = [0, 1]\).

Next, we wish to show the opposite inclusion. Suppose that \(x\) is an element of the closed interval \([0, 1]\), so \(0 \leq x \leq 1\). Then clearly \(x > -\frac{1}{n}\) and, likewise, \(x < 1 + \frac{1}{n}\) for all integers \(n\). Since this means that \(x \in A_n\) for all \(n\), we conclude that \(x\) must be in the intersection. As such, we have the inclusion \([0, 1] \subseteq \bigcap_{n=1}^{\infty} A_n\).

Since we’ve shown both inclusions, we conclude that \(\bigcap_{n=1}^{\infty} A_n = [0, 1]\).