In this document, we’re going to prove some identities using the set $B$ and the indexed sets $\{A_j\}_{j \in J}$. While we’re used to indexing over the natural numbers (and your natural inclination might be to assume that any index set is the natural numbers), here we’re actually letting our index set, $J$, be arbitrary. Because $J$ is arbitrary, you’ll notice that none of our arguments rely on any properties of $J$ - we have no idea what those might be!

First, let’s prove that $B \bigcup_{j \in J} \left( \bigcap_{j \in J} A_j \right) = \bigcap_{j \in J} (B \bigcup A_j)$.

Remember that we typically prove the equality of two sets by showing that each is a subset of the other. Here, that would mean showing that $B \bigcup \left( \bigcap_{j \in J} A_j \right) \subseteq \bigcap_{j \in J} (B \bigcup A_j)$ and that $\bigcap_{j \in J} (B \bigcup A_j) \subseteq B \bigcup \left( \bigcap_{j \in J} A_j \right)$. So our argument will naturally divide itself into two parts.

Let’s start by proving the first inclusion, $B \bigcup \left( \bigcap_{j \in J} A_j \right) \subseteq \bigcap_{j \in J} (B \bigcup A_j)$. To do so, we’ll think about tracking elements - that is, we want to show that if $x$ is an element of $B \bigcup \left( \bigcap_{j \in J} A_j \right)$, then it necessarily has to be an element of $\bigcap_{j \in J} (B \bigcup A_j)$. If we can show this, it’ll tell us that every element of $B \bigcup \left( \bigcap_{j \in J} A_j \right)$ is also an element of $\bigcap_{j \in J} (B \bigcup A_j)$, giving us the desired inclusion.

Okay, so we want to consider some arbitrary $x \in B \bigcup \left( \bigcap_{j \in J} A_j \right)$. If $x$ is in this union, then there are three possible cases: $x$ could be only in $B$ and not in $\left( \bigcap_{j \in J} A_j \right)$, $x$ could be only in $\left( \bigcap_{j \in J} A_j \right)$ and not in $B$, or $x$ could be in both $B$ and $\left( \bigcap_{j \in J} A_j \right)$. As it turns out, we only need to think about the first two cases, since proving that having either $x \in B$ or $x \in \bigcap_{j \in J} A_j$ leads to the desired inclusion.

Let’s start with the first case. If $x \in B$, then clearly $x \in (B \bigcup A_j)$ for all $j \in J$ (since we’re just adding elements). Since it’s in each $B \bigcup A_j$, we have $x \in \bigcap_{j \in J} (B \bigcup A_j)$. Next, we can consider the second case, where $x \in \bigcap_{j \in J} A_j$. In order to be in the intersection, we must have $x \in A_j$ for all $j \in J$. Again, this tells us that $x \in (B \bigcup A_j)$ for all $j$, since we’re just adding elements. As such, we have $x \in \bigcap_{j \in J} (B \bigcup A_j)$. 

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Since we’ve covered those two cases (and, implicitly, the third case!), we’ve finally got our first inclusion,

\[ B \cup \left( \bigcap_{j \in J} A_j \right) \subseteq \bigcap_{j \in J} (B \cup A_j). \]

Since we need to do the other direction now, let’s start by considering an arbitrary element \( x \) of \( \bigcap_{j \in J} (B \cup A_j) \). In order to be in this intersection, it must be true that \( x \in (B \cup A_j) \) for all \( j \in J \). So, again, this naturally gives rise to three cases: \( x \in B \) but not in all \( A_j \), \( x \) is in all \( A_j \) but not in \( B \), or \( x \) is both in \( B \) and all \( A_j \). Again, the third case will be covered by our arguments for the first two cases. So let’s start by considering the case where \( x \in B \).

Then clearly \( x \in B \cup \left( \bigcap_{j \in J} A_j \right) \), since we’re just adding elements. Alternately, we could have \( x \in A_j \) for all \( j \). Then we definitely have \( x \in \bigcap_{j \in J} A_j \), so again \( x \in B \cup \bigcap_{j \in J} A_j \).

This gives us the opposite inclusion,

\[ \bigcap_{j \in J} (B \cup A_j) \subseteq B \cup \left( \bigcap_{j \in J} A_j \right). \]

Since we’ve shown both inclusions, we get the desired set equality:

\[ B \cup \left( \bigcap_{j \in J} A_j \right) = \bigcap_{j \in J} (B \cup A_j). \]

Next, let’s prove that

\[ B \bigcap \left[ \bigcup_{j \in J} A_j \right] = \bigcup_{j \in J} (B \cap A_j). \]

Again, we’ll do this by showing that each set is a subset of the other. Let’s begin with the inclusion \( B \bigcap \left[ \bigcup_{j \in J} A_j \right] \subseteq \bigcup_{j \in J} (B \cap A_j) \). Consider an arbitrary \( x \in B \bigcap \left[ \bigcup_{j \in J} A_j \right] \). Then we know that \( x \in B \) and \( x \in \bigcup_{j \in J} A_j \), so \( x \in A_{j'} \) for at least one \( j' \in J \). Since \( x \) is an element of both \( B \) and \( A_{j'} \), we definitely know that \( x \in (B \cap A_{j'}) \), so \( x \in \bigcup_{j \in J} (B \cap A_j) \).

This gives us the inclusion

\[ B \bigcap \left[ \bigcup_{j \in J} A_j \right] \subseteq \bigcup_{j \in J} (B \cap A_j). \]

Next, let’s do the opposite inclusion. Consider some \( x \in \bigcup_{j \in J} (B \cap A_j) \). In order for \( x \) to be in that union, it must be true that \( x \in (B \cap A_{j'}) \) for at least one \( j' \in J \). Well, being in that intersection means that \( x \in B \) and \( x \in A_{j'} \). So we have \( x \in \bigcup_{j \in J} A_j \) and therefore
$x \in B \cap (\bigcup_{j \in J} A_j)$. This gives us the opposite inclusion,

$$\bigcup_{j \in J} (B \cap A_j) \subseteq B \bigcap \left( \bigcup_{j \in J} A_j \right)$$

Since we’ve shown both inclusions, we get the desired set equality:

$$B \bigcap \left( \bigcup_{j \in J} A_j \right) = \bigcup_{j \in J} (B \cap A_j).$$

Next, let’s prove that

$$B \setminus \left( \bigcup_{j \in J} A_j \right) = \bigcap_{j \in J} (B \setminus A_j).$$

Following the same structure as our previous proofs, we’ll start by considering some arbitrary $x \in B \setminus \left( \bigcup_{j \in J} A_j \right)$. This means that $x \in B$, but $x \notin \bigcup_{j \in J} A_j$. Since $x$ is not in the union of the $A_j$, clearly it can’t be in any of the individual $A_j$. So we have $x \in B \setminus A_j$ for all $j \in J$, which can be equivalently written as $x \in \bigcap_{j \in J} (B \setminus A_j)$. This gives us the inclusion

$$B \setminus \left( \bigcup_{j \in J} A_j \right) \subseteq \bigcap_{j \in J} (B \setminus A_j).$$

Next, let’s go in the opposite direction and start by considering an arbitrary $x \in \bigcap_{j \in J} (B \setminus A_j)$. Then $x \in B$ and $x \notin A_j$ for all $j \in J$. Since $x$ is not in any of the individual $A_j$, obviously $x \notin \bigcup_{j \in J} A_j$. So we have $x \in B \setminus \bigcup_{j \in J} A_j$. This gives us the inclusion

$$\bigcap_{j \in J} (B \setminus A_j) \subseteq B \setminus \left( \bigcup_{j \in J} A_j \right).$$

Since we’ve shown both inclusions, we have the desired set equality:

$$B \setminus \left( \bigcup_{j \in J} A_j \right) = \bigcap_{j \in J} (B \setminus A_j).$$

Finally, let’s show that

$$B \setminus \left( \bigcap_{j \in J} A_j \right) = \bigcup_{j \in J} (B \setminus A_j)$$
Start by considering some \( x \in B \setminus \bigcap_{j \in J} A_j \). Then \( x \in B \), but \( x \notin \bigcap_{j \in J} A_j \). So there exists at least one \( j' \in J \) such that \( x \notin A_{j'} \), which means that \( x \in B \setminus A_{j'} \) and therefore \( x \in \bigcup_{j \in J} (B \setminus A_j) \). This gives us the inclusion

\[
B \setminus \left[ \bigcap_{j \in J} A_j \right] \subseteq \bigcup_{j \in J} (B \setminus A_j)
\]

Next, consider some \( x \in \bigcup_{j \in J} (B \setminus A_j) \). In order to be in that union, we must have \( x \in B \setminus A_{j'} \) for at least one \( j' \in J \). So \( x \notin A_{j'} \) and therefore \( x \notin \bigcap_{j \in J} A_j \). So we have \( x \in B \setminus \bigcap_{j \in J} A_j \). This gives us the opposite inclusion,

\[
\bigcup_{j \in J} (B \setminus A_j) \subseteq B \setminus \left[ \bigcap_{j \in J} A_j \right].
\]

Together, these inclusions give us the set equality

\[
B \setminus \left[ \bigcap_{j \in J} A_j \right] = \bigcup_{j \in J} (B \setminus A_j)
\]