Evaluate $\int_3^6 \frac{x}{\sqrt{x-2}} \, dx$

As usual, start by identifying:

$$u = x \quad v = 2(x-2)^{1/2}$$
$$du = dx \quad dv = (x-2)^{-1/2} \, dx$$

Then we can use the definite version of the IBP formula:

$$\int_a^b uv \, du = uv \bigg|_a^b - \int_a^b v \, du$$

$$\int_3^6 \frac{x}{\sqrt{x-2}} \, dx = 2x(x-2)^{1/2} \bigg|_3^6 - \int_3^6 2(x-2)^{1/2} \, dx$$

Do this integral w/ u-sub:

$$u = x-2 \quad du = dx$$

$$\int 2(x-2)^{1/2} \, dx = \int 2u^{1/2} \, du$$

$$= 2 \cdot \frac{1}{3} u^{3/2} + C$$

$$= 2 \cdot \frac{2}{3} u^{3/2} + C$$

$$= \frac{4}{3} u^{3/2} + C$$

$$= \frac{4}{3} (x-2)^{3/2} + C$$

$$= \left[ 2(6)(6-2)^{1/2} - 2(3)(3-2)^{1/2} \right] - \frac{4}{3} \cdot (x-2)^{3/2} \bigg|_3^6$$

$$= \left[ 12 \cdot \sqrt{4} - 6 \cdot \sqrt{1} \right] - \frac{4}{3} \left[ (6-2)^{3/2} - (3-2)^{3/2} \right]$$

$$= \left[ 12 \cdot 2 - 6 \cdot 1 \right] - \frac{4}{3} \left[ 4^{3/2} - 1^{3/2} \right]$$

$$= 18 - \frac{4}{3} \left[ 4^{3/2} - 1 \right]$$

$$= 18 - \frac{4}{3} \left[ (4^{1/2})^3 - 1 \right]$$

$$= 18 - \frac{4}{3} \left[ 2^3 - 1 \right]$$

$$= 18 - \frac{4}{3} \left[ 8 - 1 \right]$$

$$= 18 - \frac{4}{3} \cdot 7$$
Note that although some steps technically require u-sub, you may be able to do those integrals without actually writing out the steps for u-substitution.

**Example** Find \( \int_1^e \ln x \, dx \)

You might remember from class that the trick here is to write \( \int_1^e \ln x \, dx = \int_1^e 1 \ln x \, dx \) and choose

\[
\begin{align*}
    u &= \ln x \\
    v &= x \\
    du &= \frac{1}{x} \, dx \\
    dv &= 1 \cdot dx
\end{align*}
\]

(Also, if we follow the LPE acronym, then it suggests that we should choose \( u = \ln x \))

Once again, we're now set up to use the definite version of the IBP formula:

\[
\begin{align*}
    \int_1^e \ln x \, dx &= X \ln x \bigg|_1^e - \int_1^e x \frac{1}{x} \, dx \\
    &= \left[ e \ln(e) - 1 \ln(1) \right] - \int_1^e 1 \, dx \\
    &= \left[ e \cdot 1 - 1 \cdot 0 \right] - x \bigg|_1^e \\
    &= e - (e - 1) \\
    &= e - e + 1 \\
    &= 1
\end{align*}
\]
Evaluate \( \int_{3}^{6} \frac{x}{\sqrt{x-2}} \, dx \)

As usual, start by identifying:

\[
\begin{align*}
    & u = x \quad v = 2(x-2)^{3/2} \\
    & du = dx \quad dv = (x-2)^{1/2} \, dx
\end{align*}
\]

Then we can use the definite version of the IBP formula:

\[
\int_{a}^{b} uv \, du = uv \bigg|_{a}^{b} - \int_{a}^{b} v \, du
\]

\[
\int_{3}^{6} \frac{x}{\sqrt{x-2}} \, dx = 2x(x-2)^{3/2} \bigg|_{3}^{6} - \int_{3}^{6} 2(x-2)^{1/2} \, dx
\]

Do this integral w/ u-sub:

\[
\begin{align*}
    & u = x-2 \quad du = dx \\
    & \int 2(x-2)^{3/2} \, dx = \int 2u^{3/2} \, du \\
    & = 2 \cdot \frac{3}{2} u^{3/2} + C \\
    & = 3 \cdot \frac{2}{3} u^{3/2} + C \\
    & = 4/3 (x-2)^{3/2} + C
\end{align*}
\]

\[
\int_{3}^{6} \frac{x}{\sqrt{x-2}} \, dx = \left[ 2(6)(6-2)^{3/2} - 2(3)(3-2)^{3/2} \right] - \frac{4}{3} \cdot (x-2)^{3/2} \bigg|_{3}^{6}
\]

\[
\begin{align*}
    &= \left[ 24 - 6 \sqrt{1} \right] - \frac{4}{3} \left[ (6-2)^{3/2} - (3-2)^{3/2} \right] \\
    &= \left[ 24 - 6 \right] - \frac{4}{3} \left[ 4^{3/2} - 1^{3/2} \right] \\
    &= 18 - \frac{4}{3} \left[ (\sqrt[3]{4})^3 - 1 \right] \\
    &= 18 - \frac{4}{3} \left[ 8 - 1 \right] \\
    &= 18 - \frac{4}{3} \cdot 7
\end{align*}
\]
\[
\begin{align*}
&= \frac{54}{3} - \frac{28}{3} \\
&= \sqrt{\frac{26}{3}}
\end{align*}
\]

Note that although some steps technically require u-sub, you may be able to do those integrals without actually writing out the steps for u-substitution.

**EXII** Find \( \int_1^e \ln x \, dx \)

You might remember from class that the trick here is to write \( \int_1^e \ln x \, dx = \int_1^e 1 \cdot \ln x \, dx \) and choose

\[
\begin{align*}
  u &= \ln x \\
  v &= x \\
  du &= \frac{1}{x} \, dx \\
  dv &= 1 \cdot dx
\end{align*}
\]

(also, if we follow the LPF acronym, then it suggests that we should choose \( u = \ln x \)!) Once again, we're now set up to use the definite version of the IBP formula:

\[
\int_1^e \ln x \, dx = x \ln x \bigg|_1^e - \int_1^e x \cdot \frac{1}{x} \, dx
\]

\[
= [e \ln (e) - 1 \ln (1)] - \int_1^e 1 \cdot dx
\]

recall that \( \ln(x) \) is the exponent that you need to raise \( e \) to in order to get \( x \).

i.e., \( e^{\ln(x)} = x \)

So \( \ln(e) = 1 \) because \( e^1 = e \)

\( \ln(1) = 0 \) because \( e^0 = 1 \)

\[
= [e \cdot 1 - 1 \cdot 0] - x \bigg|_1^e
\]

\[
= e - e + 1
\]

\[
= 1
\]
EXII (2.4.15-Curve Sketching)
Sketch f(x) = \(\frac{4}{3}x^3 - 2x^2 + x\)

1. Find f'(x), f''(x):
   
   \[
f'(x) = \frac{4}{3}x^2 - 2x + 1
   = 4x^2 - 4x + 1
   
   f''(x) = 8x - 4
   \]

2. Find local extrema (maxima/minima):
   
   \[
f'(x) = 4x^2 - 4x + 1 = 0
   \]
   
   \[
x = \frac{4 \pm \sqrt{16 - 4(4)(1)}}{2(4)}
   = \frac{4 \pm 0}{8}
   \]
   
   \[
x = \frac{1}{2}
   \]

2nd-derivative test:

   \[
f''(\frac{1}{2}) = 8(\frac{1}{2}) - 4 = 0 \text{ inconclusive!}
   \]

   so we need to fall back on the 1st derivative test

   
   \[
   \begin{array}{c|c|c|c}
   & x < \frac{1}{2} & x = \frac{1}{2} & x > \frac{1}{2} \\
   \hline
   f'(x) & - & + & +
   \end{array}
   \]

   Since f'(x) doesn't change sign, \(x = \frac{1}{2}\) is not a local extremum.

3. Find IPs:

   \[
f''(x) = 8x - 4 = 0
   \]

   \[
x = \frac{4}{8} = 1/2
   \]

   checking if \(x = \frac{1}{2}\) is an IP -

   \[
   f''(\frac{1}{2}) = 4(2x - 1)
   \]

   Since the sign of f''(x) changes at \(x = \frac{1}{2}\), this is an inflection point!

4. Find intercepts:

   x-ints: f(x) = \(\frac{4}{3}x^3 - 2x^2 + x = 0\)

   \[
x(x^2 - 2x + 1) = 0
   x = 0
   \]

   \[
x^2 - 2x + 1 = 0
   x = \frac{2 \pm \sqrt{4 - 4(1)(1)}}{2(1)}
   \]

   \[
x = 1 \pm 1
   \]

   But \(x = 1 + 1 < 0\) so this doesn't give us any more x-ints

   y-ints: f(0) = 0 = \(\frac{4}{3}0^3 - 0^2 + 0\)

   so there's an x and y intercept at (0,0).

5. Putting everything together:

   ![Graph of f(x) = \(\frac{4}{3}x^3 - 2x^2 + x\)]

   remember: not only is \(x = \frac{1}{2}\) an IP, but f'(\(\frac{1}{2}\)) = 0 so the function is briefly (instantaneously) horizontal there.

   fill in points, then draw around them/ connect the dots.
EX1 (§2.4 #20 - Optimization)

An athletic field has the shape shown below:

![Field Diagram]

The perimeter is going to be used as a 440-yard track.

Find the value of \( x \) for which the area of the shaded rectangular region of the field is as large as possible.

Start by writing equations:

\[
P = 2h + 2\pi r(\frac{h}{2}) = 2h + \pi x = 440 \text{ yards} \quad \text{(constraint)}
\]

\[
A = xh \quad \text{(objective)}
\]

D'uneing \( A \) in terms of \( x \):

\[
2h + \pi x = 440
\]

\[
h = \frac{1}{2}(440 - \pi x)
\]

So, then \( A = x \cdot \frac{1}{2}(440 - \pi x) \)

\[
= 220x - \frac{\pi}{2}x^2
\]

Now, maximizing \( A \):

\[
\frac{dA}{dx} = 220 - \frac{\pi}{2}(3)x
\]

\[
= 220 - \pi x \cdot 0
\]

\[
x = \frac{220}{\pi}
\]

\[
\frac{d^2A}{dx^2} = -\pi < 0 \quad \text{so} \quad x = \frac{220}{\pi} \text{ is a local max}
\]

But we also need to check endpoints:

Here, endpoints are \( x = 0 \) and \( x = 220 \)

\[
A(0) = 220 \cdot 0 - \frac{\pi}{2} \cdot 0^2 = 0
\]

\[
A(220) = 220 \cdot 220 - \frac{\pi}{2} \left( \frac{220}{\pi} \right)^2 < 0
\]

So, area is maximized at:

\[
x = \frac{220}{\pi} \text{ yards}
\]

\[
h = \frac{1}{2}(440 - \pi \cdot \frac{220}{\pi}) = \frac{220 - 110}{\pi} = \frac{110}{\pi} \text{ yards}
\]

EX2 (§3.3 #45 - Related Rates)

The volume of a spherical carcinous tumor is given by \( V = \frac{4}{3}\pi x^3 \), where \( x \) is the diameter of the tumor.

If the tumor is growing at a rate of 0.4 mm/day when the diameter is 10 mm, how fast is the volume changing at that time?

Recall that when we get this kind of problem, we differentiate the volume with respect to time, and the diameter is changing with time.

Plugging in values from the problem statement:

\[
\frac{dV}{dt} = \frac{4}{3}\pi \cdot (10 \text{ mm})^2 \times 0.4 \text{ mm/day}
\]

\[
= \frac{4}{3} \pi \cdot (10 \text{ mm})^2 \times 0.4 \text{ mm/day}
\]

\[
= \frac{4}{3} \pi \cdot 100 \text{ mm}^3/\text{day}
\]

EX3 (§3.3 #18 - Implicit Differentiation)

Find \( \frac{dy}{dx} \) for the curve \( x^3y + xy^2 = 4 \)

Differentiating:

\[
\frac{d}{dx} \left[ x^3y + xy^2 \right] = \frac{d}{dx} [4]
\]

\[
3x^2y + x \cdot \frac{dy}{dx} + y^2 + x \cdot 2y \cdot \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} \left( x^3 + 3xy^2 \right) = -(y^2 + 3xy)
\]

\[
\frac{dy}{dx} = \frac{-y^2 - 3xy}{x^3 + 3xy^2}
\]
Recall: A function \( f(x) \) is differentiable at \( x = a \) if the following limit exists:

\[
\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

(think: "is there a tangent line whose slope I could find?")

A function \( f(x) \) is continuous at \( x = a \) if \( \lim_{x \to a} f(x) = f(a) \). (This requires:
- \( f(a) \) is defined at \( x = a \).
- \( \lim f(x) \) exists.
- \( \lim f(x) \) and \( f(a) \) agree.)

If \( f(x) \) is differentiable at \( x = a \), then it is also continuous at \( x = a \).

Examples of discontinuities:

1. \( f(a) \) not defined
2. \( \lim_{x \to a} f(x) \) does not exist
3. Both \( f(a) \) not defined and \( \lim f(x) \) does not exist

Examples of non-differentiability:

1. No tangent line at \( x = a \).
2. Vertical tangent line (i.e., slope not defined)

Ex. (§1.5 - Differentiability & Continuity)

<table>
<thead>
<tr>
<th>( x )</th>
<th>Continuous</th>
<th>Differentiable</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>0.01</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>-2</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>2</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>-3</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>3</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Ex. (§1.4 - #38 - Limit Definition of a derivative)

Recall that \( F'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \).

Use this limit definition of the derivative to find \( f'(x) \) for \( f(x) = x + 11 \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-(x+h) + 11 - (-x + 11)}{h}
\]

\[
= \lim_{h \to 0} \frac{-x + h + 11 + x - 11}{h}
\]

\[
= \lim_{h \to 0} \frac{-h}{h}
\]

\[
eq \lim_{h \to 0} -1 = -1
\]

**NOTE:** You should always get cancellation with the \( h \) in the denominator! If not, something is wrong!
\[ \int x e^{\frac{x^2}{2}} \, dx \]

\[ u = x \quad v = 2e^{\frac{x^2}{2}} \quad \text{to find } v: \int e^{\frac{x^2}{2}} \, dx = \frac{1}{2} e^{\frac{x^2}{2}} + C \]

\[ \text{using the rule } \int e^{x^2} \, dx = C e^{x^2} + C \]

so

\[ \int x e^{\frac{x^2}{2}} \, dx = 2x e^{\frac{x^2}{2}} - \int 2 e^{\frac{x^2}{2}} \, dx \]

\[ = 2xe^{\frac{x^2}{2}} - 2 \int e^{\frac{x^2}{2}} \, dx \]

\[ = 2xe^{\frac{x^2}{2}} - 2 \cdot 2 e^{\frac{x^2}{2}} + C \]

we already did this integral to find \( C \):

\[ = 2e^{\frac{x^2}{2}} (x - 2) + C \]

---

**EX1 (§9.3 #25 - Area under a curve)**

Find the shaded area.

Recall: \[ \text{Area between } f(x) \text{ and } g(x): \int_a^b [f(x) - g(x)] \, dx \]

Here, \( f(x) = x(\sqrt{4-x^2}) \)

\[ g(x) = 0 \quad \text{(the x-axis!)} \]

Visually, we see that the "top"/"bottom" function changes at \( x = 0 \).

So we'll need two integrals.

To find the other two points of intersection:

*Note: also called x-intercepts*

\[ f(x) = g(x) \]

\[ x(\sqrt{4-x^2}) = 0 \]

\[ x = 0 \quad \sqrt{4-x^2} = 0 \]

\[ 4-x^2 = 0 \]

\[ x^2 = 4 \]

\[ x = \pm 2 \]

So, \( A = \int_{-2}^{0} [0 - x\sqrt{4-x^2}] \, dx + \int_{0}^{2} [x\sqrt{4-x^2} - 0] \, dx \)

\[ = -\int_{-2}^{0} x\sqrt{4-x^2} \, dx + \int_{0}^{2} x\sqrt{4-x^2} \, dx \]

Can save ourselves some duplicate work by first computing:

\[ \int x(\sqrt{4-x^2}) \, dx = \int x(\sqrt{4-x^2}) \, dx = \int u \cdot \frac{du}{\sqrt{u^2-a^2}} = -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \ln |u| + \frac{1}{2} a \cdot \frac{1}{2} u^{\frac{3}{2}} + C \]

\[ u = \sqrt{4-x^2} \]

\[ du = -2x \, dx \quad \text{or} \quad \frac{du}{a} = x \, dx \]

So, \( A = -\int_{-2}^{0} x(\sqrt{4-x^2}) \, dx + \int_{0}^{2} x(\sqrt{4-x^2}) \, dx \)

\[ = \left(-\frac{1}{2} x(4-x^2)^{\frac{3}{2}} \right)^{1}_{-2} + \left(-\frac{1}{2} x(4-x^2)^{\frac{3}{2}} \right)^{0}_{-2} \]

\[ = \left(\frac{1}{3} (4-x^2)^{\frac{3}{2}} \right)^{1}_{-2} + \left(-\frac{1}{3} (4-x^2)^{\frac{3}{2}} \right)^{0}_{-2} \]

\[ = \frac{1}{3} \left[ (4-0)^{\frac{3}{2}} - (4-(-2))^\frac{3}{2} \right] - \frac{1}{3} \left[ (4-2)^{\frac{3}{2}} - (4-0)^{\frac{3}{2}} \right] \]

\[ = \frac{1}{3} \left[ 4^{\frac{3}{2}} - (4+4)^{\frac{3}{2}} \right] - \frac{1}{3} \left[ (4-2)^{\frac{3}{2}} - 4^{\frac{3}{2}} \right] \]

\[ = \frac{4^{\frac{3}{2}}}{3} + \frac{4^{\frac{3}{2}}}{3} \]

\[ = \frac{2}{3} \cdot 4^{\frac{3}{2}} \]

\[ = \frac{2}{3} \left( 4^{\frac{3}{2}} \right)^3 \]

\[ = \left( \frac{16}{3} \right) \]
Zombie Apocalypse
(aka an example of an epidemic growth model)

Zombies have been spotted in Minneapolis and the CDC needs a model to predict the number of infected people at an arbitrary time $t$, measured in weeks.

Minneapolis has a population of approximately 422,000. At the beginning of the first week, there are 2,110 zombies reported. After three weeks, there are 10,550 zombies.

(a) How many zombies will there be after 2 months?

To answer this, we need to start by finding an equation that gives the number of zombies, $f(t)$, as a function of time $t$.

We can start by remembering that in an epidemic model,

$$f(t) = \frac{P}{1 + Be^{-ct}}$$

where $P$ is the total population, $B$, $c$ are positive constants.

We can use the information given in the problem statement to solve for the constants $B$ and $c$.

It tells us that:

- $f(0) = 2,110$
- $f(3) = 10,550$

Using the first point, we can solve for $B$ as:

$$f(0) = 2,110 = \frac{422,000}{1 + Be^{-c(0)}}$$

$$= \frac{422,000}{1 + Be^0}$$

$$= \frac{422,000}{1 + B}$$
We can then use the other point to solve for $c$:

$$f(3) = 10,550 = \frac{422,000}{1 + 199e^{-3c}}$$

$$10,550 (1 + 199e^{-3c}) = 422,000$$

$$1 + 199e^{-3c} = \frac{422,000}{10,550} = 40$$

$$199e^{-3c} = 40 - 1$$

$$199e^{-3c} = 39$$

$$e^{-3c} = \frac{39}{199}$$

$$\ln(e^{-3c}) = \ln(\frac{39}{199})$$

$$-3c = \ln(\frac{39}{199})$$

$$c = -\frac{\ln(\frac{39}{199})}{3}$$

$$\approx -1.22$$

So our function is

$$f(t) = \frac{422,000}{1 + 199e^{-1.22t}}$$
To find the number of zombies after 2 months, we want to compute \( f(8) \). (Note that 2 months = 8 weeks)

Doing so, we find that

\[
f(8) = \frac{422,000}{1 + 199e^{-1.22(8)}} \approx 417,208 \text{ zombies}
\]

So the city has nearly been overrun!

(b) When will Rick Grimes be the only human left?

In less flippant terms, this is asking what \( t \) will be when \( f(t) = 421,999 \), i.e., when all but one of the citizens of Minneapolis have joined the zombie horde.

To answer this, we want to solve

\[
f(t) = 421,999
\]

\[
\frac{422,000}{1 + 199e^{-1.22t}} = 421,999.
\]

\[
1 + 199e^{-1.22t} = \frac{422,000}{421,999}
\]

\[
199e^{-1.22t} = \frac{422,000}{421,999} - 1
\]

\[
e^{-1.22t} = \frac{1}{199} \left( \frac{422,000}{421,999} - 1 \right)
\]

\[
-1.22t = \ln \left[ \frac{1}{199} \left( \frac{422,000}{421,999} - 1 \right) \right]
\]

\[
t = -\frac{1}{1.22} \ln \left[ \frac{1}{199} \left( \frac{422,000}{421,999} - 1 \right) \right]
\]

\[
\approx 15 \text{ weeks}
\]
§ 27 #11:

Hamburgers at a sports arena used to cost $4 each, with an average of 10,000 hamburgers sold on each game night. When the price was raised to $4.40 each, average sales dropped to 8,000 per night.

(a) If the demand curve is linear, what price will maximize the nightly revenue from hamburgers?

The phrase "the demand curve is linear" means that the function for demand is a straight line.

How can we come up with an equation for \( p(x) \), the demand function?

Remember that: \( p(x) = \frac{\text{price}}{\text{unit}} \), so \( x \) is the # of units, or # of hamburgers sold, and \( p(x) \) is the price.

A straight line is uniquely determined by two points on the line.

The problem statement gives us two points:

\( (8000, 4.40) \) and \( (10000, 4) \)

So we can calculate the slope as

\[
m = \frac{4.40 - 4}{8000 - 10000} = \frac{0.4}{-2000} = -0.0002
\]

Be careful to be consistent with the order of your points when calculating slope, or you'll get the wrong sign!
Now, we can write an equation for \( p(x) \), using point-slope form.

\[
p(x) - 4.40 = -0.0002(x-8000) \\
= -0.0002x + 1.6 \\
\Rightarrow p(x) = -0.0002x + 6
\]

(Note: It doesn't matter what point I use – I chose to use \((8000, 4.4)\) but could have also used \((10000, 4)\) and would have gotten the same answer)

Now that we have an equation for the demand, we can write an equation for revenue.

Remember that \( R(x) \) is the amount of money earned for selling \( x \) hamburgers, and \( p(x) \) gives the price per unit sold. So if we're just reasoning using these units, we can see that

\[
\begin{bmatrix}
\text{amount earned for selling } x \\
\text{hamburgers}
\end{bmatrix} = \begin{bmatrix}
\text{amount earned} \\
\text{for selling 1 hamburger}
\end{bmatrix} \times \\
\begin{bmatrix}
\# \text{ of} \\
\text{hamburgers sold}
\end{bmatrix}
\]

\[
R(x) = p(x) \cdot x
\]

Substituting for \( p(x) \),

\[
R(x) = p(x) \cdot x = (-0.0002x + 6) \cdot x \\
= -0.0002x^2 + 6x
\]

Now, we're finally ready to maximize \( R(x) \), following our usual procedure for maximizing functions.
Taking the first derivative:
\[
\frac{dR(x)}{dx} = R'(x) = -0.0002(2)x + 6
\]
\[
= -0.0004x + 6
\]

Then, we set \( R'(x) = 0 \) and solve for \( x \) as
\[
-0.0004x + 6 = 0
\]
\[
x = \frac{-6}{-0.0004} = 15,000 \text{ hamburgers}
\]

We can then use the second derivative test to verify that this is actually a maximum:
\[
R''(x) = -0.0004 < 0 \text{ for all } x, \text{ so } x = 15,000 \text{ is definitely a maximum!}
\]

But wait—part (a) actually wanted to know what the price should be, not how many hamburgers would be being sold. So we need to go back and find \( p(15,000) \):
\[
p(x) = -0.0002x + 6
\]
\[
= -0.0002(15,000) + 6
\]
\[
= 3
\]

So the hamburgers should be priced at $3.

(b) If the fixed cost of running the food stand is $10,000 per night and it costs $0.60 to make each hamburger, what hamburger will maximize the nightly profit?
Here, we're being asked to modify our solution to (a) to account for the costs involved in running the stand and producing hamburgers.

Remember that \( P(x) = R(x) - C(x) \) is profit, which we need to work out now in order to answer the question.

Further, remember that

\[
C(x) = \text{fixed costs} + \text{variable costs}
\]

here, this is the \$10,000/night to run the stand

here, this is the \$0.60 cost to produce each hamburger.

If we make \( x \) hamburgers, this will be \( 0.60x \)

So, \( C(x) = 10,000 + 0.6x \)

Using the equation for \( R(x) \) that we found in (a), we can substitute and find an expression for \( P(x) \):

\[
P(x) = R(x) - C(x)
= (-0.0002x^2 + 6x) - (10,000 + 0.6x)
= -0.0002x^2 + 6x - 10,000 - 0.6x
= -0.0002x^2 + 5.4x - 10,000
\]

Now, we can find the value of \( x \) that maximizes \( P(x) \) like we would for any other function.

\[
\frac{dP}{dx} = P'(x) = -0.0004x(2) + 5.4
= -0.0004x + 5.4
\]
Setting $P'(x) = 0$ and solving for $x$,

$$-0.0004x + 5.4 = 0$$

$$x = \frac{-5.4}{-0.0004}$$

$$= 13,500$$

Again, we can verify that this value of $x$ maximizes $P(x)$ by using the second derivative test:

$$P''(x) = -0.0004 < 0$$ for all $x$, so we definitely have a maximum at

$$x = 13,500$$

Again, the problem actually asked for the price we should set, not for how many hamburgers would be sold. So we can go back and calculate

$$p(x) = -0.0002x + b$$

Notice this is a lowercase $p$, so we're looking at the demand function, NOT the profit function.

$$p(13,500) = -0.0002(13,500) + b$$

$$= 3.3$$

So we should price the hamburgers at $\$3.30$ each to maximize our nightly profit.
As an example of the 1st derivative test, let's do \( f(x) = x^3 + 6x^2 - 9x + 1 \).

We want to find its critical points and identify if they are maxima or minima using the 1st derivative test.

We start by taking the derivative of \( f(x) \):

\[
f'(x) = -3x^2 + 6x - 9
= -3x^2 + 6x - 9
= -3(x^2 - 4x + 3)
\]

We then set \( f'(x) = 0 \), to find the critical points.

\[
f'(x) = -3(x^2 - 4x + 3) = 0
\]

\[
x^2 - 4x + 3 = 0
\]

\[
(x - 3)(x - 1) = 0
\]

So \( x = 3 \) or \( x = 1 \)

To determine if \( x = 3, x = 1 \) are where maxima/minima occur, we want to look at what happens to the sign of \( f'(x) \) around these points.

We can do so by making what your book calls a variation chart, where we find the sign of \( f'(x) \) on various intervals. To make it easier to find the sign of \( f'(x) \), we include rows for each factor of \( f'(x) \) - here, that's \( (x - 3), (x - 1), \) and \(-3\). We can then find the sign of each factor and use those to find the overall sign of \( f'(x) \).
We find individual entries in this chart by testing values of $x$ in the given interval.

For example, we could find that $x-1$ is positive on the interval $1 < x < 3$ by choosing $x=2$ (or any number in the interval) and computing

$$x-1 = 2-1 = 1 > 0$$

To find the sign of $f'(x)$ on a given interval, we find the signs of each of its factors on the interval and "multiply" the signs.

For example, let's look at $1 < x < 3$ again. On this interval,

- $-3 < 0$
- $x-1 > 0$
- $x-3 < 0$

So $f'(x) = -3(x-1)(x-3) = (-)(+)(-) = +$

*two negative #s give you a positive*
Now that we have the chart, we can use it to decide what type of extreme points we have at $x = 1$, $x = 3$.

For $x = 1$: We have $f'(x) < 0$, so it's a local min

For $x = 3$: We have $f'(x) > 0$, so it's a local max

Finally, we should plug $x = 1$ and $x = 3$ into $f(x)$ so we can give the actual points rather than just the $x$-coordinate.

\[
f(x) = -x^3 + \frac{6}{4}x^2 - 9x + 1
\]

\[
f(1) = -(1^3) + \frac{6}{4}(1^2) - 9(1) + 1
\]
\[= -1 + 1.5 - 9 + 1
\]
\[= -3
\]

\[
f(3) = -3^3 + \frac{6}{4}3^2 - 9\cdot 3 + 1
\]
\[= -27 + 54 - 27 + 1
\]
\[= 1
\]

So we have a local minimum at $(1, -3)$ and a local maximum at $(3, 1)$.
As an aside—why would we ever want to use a first derivative test when we've seen that the second derivative test is so much faster?

Well—what if we don't have information about the second derivative?

Maybe our function $f(x)$ has a first derivative $f'(x)$ that's non-differentiable.

Maybe we're working with real-world data, so we have (for example) graphs of the position and velocity of a car, but no actual function to differentiate.

Usually, though, we'd want to use the second derivative test whenever we could.

It's also possible for the second derivative test to be inconclusive at $x = a$, if $f''(a) = 0$.

Then, we would have to fall back on the first derivative test.