The Fundamental Theorem of Tropical Geometry

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Abstract

This paper focuses on understanding the statement of the fundamental theorem of tropical geometry through concrete examples. In order to do so, we again use examples to explicitly build up the necessary background knowledge, including: tropicalization, valuations, tropical hypersurfaces, splittings, initial forms, and tropical varieties. We finish by briefly commenting on the overall structure of the proof of the fundamental theorem.

Tropical geometry is a relatively recent field. We might think of it as a variant on algebraic geometry, where we choose to work over the tropical semi-ring, but it also has strong connections to other fields like combinatorics, enumerative and real algebraic geometry, mathematical physics, number theory, symplectic geometry, and (surprisingly!) computational biology, among others. In this paper, we will be most concerned with its connections to commutative algebra, as our main focus will be on understanding the statement of the fundamental theorem of tropical algebraic geometry - which will require an understanding of topics like tropical varieties.

In that vein, we need to start by building the requisite background for the fundamental theorem. Throughout, we will work in the tropical semi-ring \( (\mathbb{R} \cup \{\infty\}, \oplus, \odot) \), where \( \oplus \) and \( \odot \) are defined as
\[
x \oplus y := \min(x, y) \quad \text{and} \quad x \odot y := x + y.
\]
These operations have all the nice properties that we might expect - both are associative and commutative, have identity elements (i.e., \( x \oplus \infty = x \) and \( x \odot 0 = x \) for all \( x \in \mathbb{R} \)), and obey a distributive law. As one might expect, it’s possible to construct and study polynomials in the tropical semi-ring.

Let \( x_1, \ldots, x_n \) be elements of \( (\mathbb{R} \cup \{\infty\}, \oplus, \odot) \). A tropical monomial in these elements is any product of \( x_1, \ldots, x_n \), where repetition is allowed, an example being:
\[
x_1 \odot x_2 \odot x_1 \odot x_4 \odot x_3 \odot x_2 \odot x_1 = x_1^3 x_2^2 x_3 x_4
\]
(Note that because \( \odot \) is commutative, we can collect like terms and use our usual product shorthand). If we evaluated this example using classical arithmetic, it would become
\[
x_1 + x_1 + x_1 + x_2 + x_2 + x_3 + x_4 = 3x_1 + 2x_2 + x_3 + x_4,
\]
which we recognize as a linear function from \( \mathbb{R}^4 \) to \( \mathbb{R} \). More generally, any tropical monomial is a linear function from \( \mathbb{R}^n \) to \( \mathbb{R} \). We can construct tropical polynomials as finite linear combinations of tropical monomials. For example,
\[
ax_1^2 x_2 \oplus bx_3 x_4 \oplus cx_1 x_4
\]
where \( a, b, c \in \mathbb{R} \) is a tropical polynomial mapping \( \mathbb{R}^4 \) to \( \mathbb{R} \). Note that this map is not necessarily linear, though; if we translate to classical arithmetic, this tropical polynomial is equivalent to writing
\[
\min\{a + 2x_1 + x_2, b + x_3 + x_4, c + x_1 + x_4\},
\]
More generally, any tropical polynomial is classically equivalent to taking the minimum of a finite set of linear functions. An example is shown in Figure 1.
Another useful notion is that of tropicalization, where we pass classical functions to the tropical semi-ring. Tropicalization allows us to reframe problems in the language of tropical geometry. Since the notion of tropicalization is not entirely straightforward, we need to begin by establishing some background material.

First, we need to introduce valuations. Let $K$ be a field, with $K^*$ denoting the non-zero elements of $K$. Then a valuation on $K$ is defined as a functional $\text{val} : K \to \mathbb{R} \cup \{\infty\}$ such that: $\text{val}(a) = \infty$ if and only if $a = 0$; $\text{val}(ab) = \text{val}(a) + \text{val}(b)$; and $\text{val}(a + b) \geq \min\{\text{val}(a), \text{val}(b)\}$ for all $a, b \in K^*$. We typically use $\Gamma_{\text{val}}$ to denote the image of $\text{val}$.

To illustrate the idea of a valuation, consider the field of Puiseux series, $k\{\{t\}\}$ - a generalization of power series which allows exponents of indeterminates to be negative and rational - with coefficients in an algebraically closed field of characteristic zero. Elements of $k\{\{t\}\}$ have the form $c(t) = c_1t^{a_1} + c_2t^{a_2} + c_3t^{a_3} + \cdots$ where $c_i \in k^*$ and $a_1 < a_2 < a_3 < \cdots$ are a strictly increasing set of rational numbers with a common denominator. We can then define the natural valuation on $k\{\{t\}\}$ as $\text{val} : k\{\{t\}\} \to \mathbb{R}$, where val maps $c(t) \in k\{\{t\}\}$ to the lowest exponent in its series representation (in our expression for $c(t)$, this would be $a_1$). To get a feel for how this valuation works, one might verify the following pair of examples:

\[
\text{val}\left(\frac{6t^4 + 7t^2}{3t}\right) = \text{val}\left(2t^3 + \frac{7}{3}t\right) = 1,
\]
\[
\text{val}(7) = 0.
\]

Here we have $\Gamma_{\text{val}} = \mathbb{Q}$, since we can construct an element $c(t) \in k\{\{t\}\}$ with lowest exponent corresponding to any given rational number. Although we will work more generally with a generic field $K$, the field of Puiseux series appears with some frequency in the study of tropical geometry.

Now that we’ve established the concept of a valuation, we’re ready to explore the tropicalization of a polynomial. Let $f(x) = \sum c_u x^u$ be a Laurent polynomial in $K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then its tropicalization, $\text{trop}(f) : \mathbb{R}^{n+1} \to \mathbb{R}$ is given by

\[
\text{trop}(f)(w) = \min\{\text{val}(c_u) + w \cdot u : u \in \mathbb{N}^{n+1}, c_u \neq 0\},
\]
which is obtained from \( f(x) \) by sending constant coefficients to their valuation and then replacing classical addition and multiplication with the tropical operations.

As an example, let \( K \) be the field of Puiseux series with real coefficients and consider

\[
f(x) = (5t^{-3/2} + 7t^{3} + 9t^{8})x_{2}^{2} + (7t^{4} + 10t^{11})x_{2}^{-3}
\]

in \( K[x_{1}^{\pm 1}, x_{2}^{\pm 1}] \). Using the previously defined natural valuation on the Puiseux series, we observe that

\[
trop(f)(w) = \min\{\text{val}(5t^{-3/2} + 7t^{3} + 9t^{8}) + w_{1} + w_{1}, \text{val}(7t^{4} + 10t^{11}) + w_{2}^{-1} + w_{2}^{-1} + w_{2}^{-1}\}
\]

\[
= \min\{-3/2 + w_{1} + w_{1}, 4 + w_{2}^{-1} + w_{2}^{-1} + w_{2}^{-1}\}
\]

\[
= \min\{-3/2 + 2w_{1}, 4 + 3w_{2}^{-1}\},
\]

for \( w = (w_{1}, w_{2}) \in \mathbb{R}^{2} \).

We already have a notion of the classical algebraic variety of a Laurent polynomial \( f \in K[x_{1}^{\pm 1}, x_{2}^{\pm 1}, \ldots, x_{n}^{\pm 1}] \) as a hypersurface of \((K^{*})^{n}\), written \( V(f) = \{ x \in (K^{*})^{n} : f(x) = 0 \} \). The same polynomial \( f \in I \) also has an associated tropical hypersurface, denoted as \( \text{trop}(V(f)) \), which consists of the set of points where the minimum of \( \text{trop}(f) \) is not unique. In order to rigorously define this, we need to first introduce the idea of initial forms and splittings.

A splitting \( \phi : \Gamma_{\text{val}} \to K^{*} \) is a group homomorphism sending \( u \mapsto t^{u} \) such that \( \text{val}(t^{u}) = u \). In our running example, \( K = k\{\{t\}\} \) and \( \text{val} : c(t) \mapsto \) lowest exponent of \( c(t) \), a natural choice of splitting is \( u \mapsto t^{u} \). If we instead let \( K = \mathbb{C} \) and use the trivial valuation (\( \text{val}(a) = 0 \) for all \( a \neq 0 \)), a natural choice might be \( 0 \mapsto 1 \). Let \( R = \{ a \in K : \text{val}(a) \geq 0 \} \) be the valuation ring of \( K \), which is local and has maximal ideal \( m = \{ a \in K : \text{val} > 0 \} \cup \{0\} \). Then let \( k = R/m \) be the resulting residue field. For \( K = k\{\{t\}\} \), we have \( R = \bigcup_{n \geq 1} k[[t^{1/n}]] \) and \( k = k \).

For \( a \) in the valuation ring \( R \), we denote its image in \( k \) as \( \overline{a} \). Similarly, for a polynomial \( f \) with coefficients in \( R \), we use \( f^{\overline{u}} \) to denote the polynomial formed by replacing each coefficient with its image in the residue field.

The initial form of \( f \) is defined relative to some fixed \( w \in (\Gamma_{\text{val}})^{n+1} \). If we let \( W = \text{trop}(f)(w) = \min\{\text{val}(c_{u}) + w \cdot u : c_{u} \neq 0\} \), then we can define the initial form of \( f \) relative to \( w \) as

\[
in_{w}(f) = \frac{f^{\overline{w}}(t^{w_{0}}x_{0}, \ldots, t^{w_{n}}x_{n})}{t^{-W} \sum_{u \in \mathbb{N}^{n+1}} c_{u}t^{w \cdot u}x^{u}}
\]

\[
= \frac{\sum_{u \in \mathbb{N}^{n+1}} c_{u}t^{w \cdot u - W}x^{u}}{\sum_{u \in \mathbb{N}^{n+1}} c_{u}t^{-\text{val}(c_{u})}x^{u}}
\]

\[
= \frac{\sum_{u \in \mathbb{N}^{n+1}} c_{u}t^{-\text{val}(c_{u})}x^{u}}{\sum_{u \in \mathbb{N}^{n+1}} c_{u}t^{-\text{val}(c_{u})}x^{u}}
\]

As an example, consider \( f = (t + t^{2})x_{0} + 2t^{2}x_{1} + 3t^{4}x_{2} \in C\{\{t\}\}[x_{0}^{\pm 1}, x_{1}^{\pm 1}, x_{2}^{\pm 1}] \) and let \( w = (2, 1, 0) \). Then

\[
W = \min\{\text{val}(t + t^{2}) + 2, \text{val}(2t^{2}) + 1, \text{val}(3t^{4}) + 0\} = \min\{3, 3, 4\} = 3,
\]

\[
in_{(2,1,0)}(f) = \frac{f^{\overline{w}}(t^{2}x_{0}, tx_{1}, x_{2})}{t^{-3}((t + t^{2})^{2}x_{0} + (2t^{2})tx_{1} + 3t^{4}x_{2})} = \frac{t^{-3}((t + t^{2})^{2}x_{0} + (2t^{2})tx_{1} + 3t^{4}x_{2})}{t^{-3}(t + t^{2})^{2}x_{0} + (2t^{2})tx_{1} + 3t^{4}x_{2}} = \frac{t^{3}x_{0} + 2t^{2}x_{0} + 3t^{4}x_{2}}{t^{3}x_{0} + 2t^{2}x_{1} + 3t^{4}x_{2}} = x_{0} + 2x_{1}.
\]
More formally, the tropical hypersurface associated with $f$ is the closure (in the Euclidean topology) in $\mathbb{R}^n$ of $\{ w \in \Gamma_{\text{val}} : \langle \text{in}_w(f) \rangle \neq \langle 1 \rangle \}$, i.e. the set of weight vectors for which $\text{in}_w(f)$ isn’t a unit in $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Tropicalization of functions allows us to elegantly define tropical varieties. Let $I$ be an ideal in $K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ with algebraic variety $X = V(I)$. Then the tropicalization of $X$, $\text{trop}(X)$, is given by the intersection of the tropical hypersurfaces defined by each $f \in I$. I.e.,

$$\text{trop}(X) = \bigcap_{f \in I} \text{trop}(V(f))$$

An important and perhaps not immediately obvious thing to note is that it’s not sufficient to just intersect the hypersurfaces corresponding to some generating set for $I$. Doing so will yield a tropical prevariety, but not necessarily a tropical variety.

At this point, we’ve introduced quite a few ways to think about objects in tropical geometry, so one might be naturally curious about how all these ideas and definitions are connected. The Fundamental Theorem of Tropical Geometry gives us a nice way to understand how many of the definitions we’ve introduced simply give us alternate ways to talk about the same type of thing. The statement of the theorem is as follows:

**Fundamental Theorem:** Let $I$ be an ideal in $K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $X = V(I)$ be its classical variety in the algebraic torus $T^n \cong (K^*)^n$. The following three subsets of $\mathbb{R}^n$ coincide:

(a) the tropical variety $\text{trop}(X)$;

(b) the closure in $\mathbb{R}^n$ of $\{ w \in \Gamma_{\text{val}} : \langle \text{in}_w(f) \rangle \neq \langle 1 \rangle \}$; and

(c) the closure of $\text{val}(X) := \{ (\text{val}(x_1), \ldots, \text{val}(x_n)) : (x_1, \ldots, x_n) \in X \}$

The proof of the theorem is too long to include in this paper (for an excellent exposition, see [2]), so we’ll instead explore the theorem by considering an example. Because the tropical variety of an ideal $I$ is given by the intersection of the tropical hypersurfaces corresponding to each $f \in I$, not the intersection of the hypersurfaces corresponding to some generating set for $I$, computing tropical varieties by hand is quite laborious. As such, many mathematicians make use of gfan (software created by Jensen, [1], for computing Gröbner bases and tropical varieties) in doing so.
Because we want to show our computations for illustrative purposes, we’ll manage the difficulty of computing tropical varieties by hand by following a relatively simple example from [3]. Let $K = \mathbb{C}\{\{t\}\}$ and $X = V(x+y+1) \subseteq (K^*)^2$. Some simple algebraic manipulation tells us that
\[
X = \{(x,y) \in (K^*)^2 : x + y + 1 = 0\} = \{(a, -a-1) : a \in K^*\}.
\]
If we tropicalize $f = x + y + 1$, we get $\text{trop}(f) = \min\{x, y, 0\}$, shown in Figure 2(a). This is the set in part (a) of the Fundamental Theorem. Next, we might consider $\text{val}(X)$. To do so, we observe that
\[
(\text{val}(a), \text{val}(-a-1)) = \begin{cases} 
(\text{val}(a), 0) & \text{if } \text{val}(a) > 0 \\
(\text{val}(a), \text{val}(a)) & \text{if } \text{val}(a) < 0 \\
(0, \text{val}(a+1)) & \text{if } \text{val}(a+1) > 0 \\
(0,0) & \text{else}.
\end{cases}
\]
So now if we look at $\text{val}(X) = \{(\text{val}(a), \text{val}(-a-1) : (a, -a-1) \in X\}$ as a whole, shown in Figure 2(b), we see that it coincides with $\text{trop}(X)$. So we observe the equivalence of subsets (a) and (c) in this example. Finally, let’s consider the subset from (b), the closure in $\mathbb{R}^2$ of $\{w \in \Gamma_{\text{val}} : \text{in}_w(1) \neq (1)\}$.

Let $w = (w_1, w_2) \in \Gamma_{\text{val}}^2$. Since the coefficient of every term of $f$ is mapped to zero under the natural valuation on $\mathbb{C}\{\{t\}\}$, we get $W = \min\{w_1, w_2, 0\}$ and
\[
\text{in}_w(f) = t^{-\min\{w_1, w_2, 0\}}(t^w_1 x + t^w_2 y + 1),
\]
which gives us
\[
\text{in}_w(f) = \begin{cases} 
(w_2 - w_1)y - w_1 & \text{if } w_1 = \min\{w_1, w_2, 0\} \\
(w_1 - w_2)x - w_2 & \text{if } w_2 = \min\{w_1, w_2, 0\} \\
w_1 x + w_2 y & \text{if } 0 = \min\{w_1, w_2, 0\}.
\end{cases}
\]
Note that if we have any degeneracy (i.e., $w_1 = w_2, w_1 = 0$, etc), $\text{in}_w(f)$ will remain well-defined. Now, we want to determine when $\text{in}_w(f)$ is not a unit in $K[[x^{\pm 1}, y^{\pm 1}]]$. This occurs exactly when $\min\{w_1, w_2, 0\} = w_1 = w_2$, when $\min\{w_1, w_2, 0\} = w_1 = 0$, or when $\min\{w_1, w_2, 0\} = w_2 = 0$, shown in Figure 2(c). So we observe that in this example, each of the three subsets is equivalent, as expected.

Although we won’t give a full proof of the Fundamental Theorem, we will make some comments about the overall flavor of the proof given in [2]. Because Kapranov’s Theorem already gives us the desired set equalities for hypersurfaces, the essential work is in generalizing from that case to arbitrary varieties. This generalization occurs via projection to the hypersurface case, so the set up for the proof focuses on proving the existence of a sufficiently nice projection map, such that the projection of any subvariety is Zariski closed and has the same dimension as the original subvariety. The proof then uses this projection map and some dimensional arguments to directly prove the necessary set containments.

One way to get a sense of the relationship between commutative algebra and tropical geometry is to consider their respective relationships with algebraic geometry. Many of the ideas in commutative algebra - ranging from Krull dimension theory to primary decompositions - are more algebraically framed versions of notions that appear in algebraic geometry. Because tropical geometry can be thought of as a variant of algebra geometry where we choose to work over the tropical semi-ring, the relationship between commutative algebra and algebraic geometry could allow us to reframe some commutative algebra problems and apply tools from tropical geometry.

References

