Theta Basis for Generalized Cluster Algebras

Elizabeth Kelley
(joint work with Man-Wai Cheung and Gregg Musiker)

December 9, 2020
A general roadmap

generalized cluster algebras

scattering diagram construction

mutation invariance, wall-crossing, and path-ordered products

building $\mathcal{A}$ from the cluster scattering diagram

broken lines, theta functions

theta basis
Ordinary Cluster Algebras

Ordinary cluster algebras were defined by Fomin and Zelevinsky in 2001.
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A cluster algebra is a type of commutative algebra whose generators, i.e. \textit{cluster variables}, are related by standard \textit{exchange relations}. The cluster variables appear in fixed size subsets called \textit{clusters}, each of which suffices to generate the entire algebra. The clusters are related via \textit{mutation}.
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A cluster algebra is specified by a set of *seed data* $\Sigma = (x, y, B)$ where

- $x = (x_1, \ldots, x_n)$ is the *initial cluster*,
- $y = (y_1, \ldots, y_n)$ is the *initial coefficient cluster*,
- and the *exchange matrix* $B$ is an $n \times n$ skew-symmetrizable matrix with entries in $\mathbb{Z}$. 

Ordinary Cluster Algebras

We can picture the structure

\[
\begin{array}{c}
(x', y', B') \\
\downarrow \mu_1 \\
(x^{(n)}, y^{(n)}, B^{(n)}) \\
\downarrow \mu_n
\end{array}
\]

\[
\begin{array}{c}
(x, y, B) \\
\downarrow \mu_2 \\
(x'', y'', B'') \\
\downarrow \mu_3
\end{array}
\]

\[
\begin{array}{c}
(x''', y''', B''') \\
\downarrow \cdots
\end{array}
\]

where \( \mu_k \) stands for mutation in direction \( k \).
Algebraically, mutation in direction $k$ is an involutive operation which maps a seed $\Sigma = (x, y, B)$ to another seed $\mu_k(\Sigma) = (x', y', B')$ via the relations:

$\begin{align*}
    b'_{ij} &= \begin{cases} 
        -b_{ij} & \text{if } i = k \text{ or } j = k \\
        b_{ij} + [-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+ & \text{else}
    \end{cases} \\
    y'_i &= \begin{cases} 
        y_k^{-1} & \text{if } i = k; \\
        y_i y_k[b_{ki}]_+ (1 \oplus y_k)^{-b_{ki}} & \text{if } i \neq k
    \end{cases} \\
    x'_i &= \begin{cases} 
        x_i & \text{if } i \neq k \\
        \frac{y_k \prod x_j^{[b_{ik}]_+} + \prod x_j^{-[b_{ik}]_-}}{x_k (1 \oplus y_k)} & \text{if } i = k
    \end{cases}
\end{align*}$

where $[a]_+ = \max(0, a)$.

Note that these exchange relations are always binomial.
Example: Let $\Sigma = \left( (x_1, x_2), (y_1, y_2), \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$ ("Type A2")
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Suppose I want to find $\mu_1(\Sigma)$. Then I compute

$$B' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$y_1' = \frac{1}{y_1}$$

$$y_2' = y_1y_2$$

$$x_1' = \frac{y_1 + x_2}{x_1}$$

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So $\mu_k(\Sigma) = \left( (\frac{y_1 + x_2}{x_1}, x_2), (\frac{1}{y_1}, y_1 y_2), \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$
If we continue, alternating the direction of mutation, we find that this seed generates a cluster algebra with cluster variables

\[
\left\{ x_1, x_2, \frac{1 + x_1 y_2}{x_2}, \frac{y_1 + x_2}{x_1}, \frac{y_1 + x_2 + x_1 y_1 y_2}{x_1 x_2} \right\}
\]

Notice: Each cluster variable is a Laurent polynomial in \(x_1, \ldots, x_n\). In fact, the cluster variables can be written as Laurent polynomials in terms of any choice of initial cluster. ("Laurent phenomenon") Moreover, the coefficients of these Laurent polynomials are all positive integers. ("positivity") These observations lead to a natural question: how can we define a "good" basis for a cluster algebra?
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Ordinary Cluster Algebras

There are many known bases for various subclasses:

- Monomial basis for Dynkin-type cluster algebras (Caldero-Keller)
- Atomic basis for types $A$ and $\tilde{A}$ (Sherman-Zelevinsky, Irelli, Dupont-Thomas)
- Bracelet basis for surface type (Musiker-Schiffler-Williams)
- Bangle basis for surface type (Musiker-Schiffler-Williams)
- Band basis for surface type (Thurston)
- Standard monomial basis for quantum acyclic algebras (Berenstein-Fomin-Zelevinsky)
- Triangular basis for quantum acyclic algebras (Berenstein-Zelevinsky)
- Dual canonical basis for quantum acyclic algebras (Nakajima, Kimura-Qin)
- Generic basis for quantum acyclic algebras (Dupont, Plamondon)
- Greedy basis for quantum rank 2 cluster algebras (Lee-Li-Rupel-Zelevinsky)
- Theta basis for acyclic cluster algebras (Gross-Hacking-Keel-Kontsevich)
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Generalized Cluster Algebras

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- and $a = (a_{i,s})_{i \in [n], s \in [r_i-1]}$ is a collection of scalars such that the coefficients of the $i^{th}$ exchange polynomial are determined by the tuple $(a_{i,0}, a_{i,1}, \ldots, a_{i,r_i-1}, a_{i,r_i})$. 
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We focus on a subtype of generalized cluster algebras called \textit{reciprocal generalized cluster algebras}.
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In these algebras, we require that:

- \( a_{i,1} = a_{i,r_i} = 1 \).
- and \( a_{i,k} = a_{i,r_i-k} \) for all \( 2 \leq k \leq r_i - 1 \).
We focus on a subtype of generalized cluster algebras called *reciprocal generalized cluster algebras*. In these algebras, we require that:

- $a_{i,1} = a_{i,r_i} = 1$.
- and $a_{i,k} = a_{i,r_i-k}$ for all $2 \leq k \leq r_i - 1$.

The data $r_i$ and $(a_{i,s})$ specify an exchange polynomial of the form $1 + a_{i,1}u + \cdots + a_{i,r_i-1}u^{r_i-1} + u^{r_i}$. The reciprocal condition gives us the identity

$$1 + a_{i,1}u + \cdots + a_{i,r_i-1}u^{r_i-1} + u^{r_i} = 1 + a_{i,r_i-1}u + \cdots + a_{i,1}u^{r_i-1} + u^{r_i}$$
Here, mutation in direction $k$ is defined by the exchange relations:

\[
\begin{align*}
    b'_{ij} &= \begin{cases} 
        -b_{ij} & i = k \text{ or } j = k \\
        b_{ij} + r_k \left( [-b_{ik} + b_{kj} + b_{ik}b_{kj}]_+ \right) & i, j \neq k 
    \end{cases} \\
    y'_i &= \begin{cases} 
        y_{ki}^{-1} & i = k \\
        y_i \left( y_k^{[b_{ki}]} \right)^{r_k} \left( 1 \oplus a_{k,1}y_k \oplus \cdots \oplus a_{k,r_k-1}y_k^{r_k-1} \oplus y_k^{r_k} \right)^{-b_{ki}} & i \neq k 
    \end{cases} \\
    x'_i &= \begin{cases} 
        x_{ki}^{-1} \left( \prod_{j=1}^n x_j^{[-b_{jk}]} \right)^{r_k} \frac{1 + a_{k,1}y_k + \cdots + y_k^{r_k}}{1 \oplus a_{k,1}y_k + \cdots \oplus a_{k,r_k-1}y_k^{r_k-1} \oplus y_k^{r_k}} & i = k \\
        x_i & i \neq k 
    \end{cases} \\
    a'_{k,i} &= a_{k,r_k-i}
\end{align*}
\]

where $\hat{y}_k = y_k x_1^{b_{1k}} \cdots x_n^{b_{nk}}$. 
Note that when all \( r_i = 1 \), we recover the ordinary exchange relations:

\[
\begin{align*}
    b'_{ij} &= \begin{cases} 
        -b_{ij} & \text{if } i = k \text{ or } j = k \\
        b_{ij} + 1 \left( [-b_{ik}]_+ + b_{kj} + b_{ik}b_{kj} \right) & \text{if } i, j \neq k
    \end{cases} \\
    y'_i &= \begin{cases} 
        y_k^{-1} & \text{if } i = k \\
        y_i \left( [y_k^{-1}]_+ \right) \left( 1 \oplus y_k \right)^{-1} b_{ki} & \text{if } i \neq k
    \end{cases} \\
    x'_i &= \begin{cases} 
        x_k^{-1} \left( \prod_{j=1}^n x_j \right) \left( 1 \oplus y_k \right)^{-1} \frac{1 + y_k}{1 + y_k} & \text{if } i = k \\
        x_i & \text{if } i \neq k
    \end{cases}
\end{align*}
\]
Example: Let $\mathcal{A} = \left( x, y, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, (3, 1), ((1, a, a, 1), (1, 1)) \right)$
**Example:** Let \( \mathcal{A} = \left( x, y, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, (3, 1), ((1, a, a, 1), (1, 1)) \right) \)

Then mutating at \( k = 1 \) gives

\[
x_1' = \frac{x_2^3(1 + ay_1x_2^{-1} + ay_1x_2^{-2} + x_2^{-3})}{x_1}
\]

\[
= \frac{x_2^3 + ay_1x_2^2 + ay_1x_2 + 1}{x_1}
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As before, we could compute all cluster variables using a sequence of mutations which alternates between $k = 1$ and $k = 2$. 
Doing so, we find

\[
\begin{align*}
\theta_1, \theta_2, & \quad \frac{1+2x_1+ax_2+x_1^2+ax_1x_2+ax_2^2+x_2^3}{x_1x_2^2}, \quad \frac{1+ax_2+ax_2^2+x_2^3}{x_1}, \quad \frac{1+x_1+ax_2+ax_2^2+x_2^3}{x_1x_2}, \\
\frac{1+x_1}{x_2}, & \quad \frac{1+3x_1+ax_2+3x_1^2+2ax_1x_2+ax_2^2+x_2^3+x_1^2}{x_1x_2^3}, \\
x_1^2x_2 & \quad \left(1 + 3x_1 + 2ax_2 + 3x_1^2 + 4ax_1x_2 + 2ax_2^2 + 2x_2^3 + x_1^3 + a^2x_2^2 + 2ax_1^2x_2 + 3ax_1x_2^2 + 3x_1x_2^3 + a^2x_1x_2^3 + 2ax_2^4 + ax_1^2x_2^2 + a^2x_1x_2^3 + ax_1x_2^4 + a^2x_2^4 + 2ax_2^5 + x_2^6\right)
\end{align*}
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Generalized cluster algebras exhibit many of the structural properties of ordinary cluster algebras, including the Laurent Phenomenon.
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Although Chekhov and Shapiro conjectured that positivity should hold for arbitrary generalized cluster algebras, their conjecture has only been verified for certain subclasses.
Generalized Cluster Algebras

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Although Chekhov and Shapiro conjectured that positivity should hold for arbitrary generalized cluster algebras, their conjecture has only been verified for certain subclasses.

Likewise, bases are only known for particular subclasses.
Known bases:

- Greedy basis for rank 2 generalized cluster algebras (Rupel)
- Monomial basis for acyclic generalized cluster algebras (Bai-Chen-Ding-Xu)
- Bracelet basis for orbifold type (Felixson-Tumarkin)
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Objective: A theta basis for generalized cluster algebras.
From a geometric point of view, we study cluster algebras by studying *cluster varieties*.
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From a geometric point of view, we study cluster algebras by studying *cluster varieties*. Cluster varieties appear in pairs \((A, X)\) called *cluster ensembles*.

- The \(A\)-variety encodes information about the cluster variables \((x_i)\).
- The \(X\)-variety encodes information about the coefficients \((y_i)\).

For the special case of cluster algebras with principal coefficients, we can consider the \(A\) *cluster variety with principal coefficients*, denoted \(A_{\text{prin}}\).
Scattering Diagrams

Scattering diagrams are a powerful tool from algebraic geometry which can be used to study the structure of cluster algebras.

Gross, Hacking, Keel, and Konstevich used cluster scattering diagrams to:

- Define theta functions and prove that they give a canonical positive basis for cluster algebras.
- Give a proof of positivity for arbitrary cluster algebras.

A scattering diagram is a collection of walls $d$ and automorphisms $f_d$. Each wall is a codimension-1 cone (in rank 2, these are simply lines) and the automorphisms are formal power series in $z$. 

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Scattering Diagrams for Generalized Cluster Algebras

To adapt the cluster scattering diagram construction of Gross, Hacking, Keel, and Kontsevich for the generalized setting, we begin by defining the *generalized fixed data* $\Gamma$:

- The cocharacter lattice $N$ with skew-symmetric bilinear form $\{\cdot, \cdot\} : N \times N \to \mathbb{Q}$.
- A saturated sublattice $N_{uf} \subseteq N$ called the *unfrozen sublattice*.
- An index set $I$ with $|I| = \text{rank}(N)$ and subset $I_{uf} \subseteq I$ such that $|I_{unf}| = \text{rank}(N_{uf})$.
- A set of positive integers $\{d_i\}_{i \in I}$ such that $\gcd(d_i) = 1$.
- A sublattice $N^\circ \subseteq N$ of finite index such that $\{N_{uf}, N^\circ\} \subseteq \mathbb{Z}$ and $\{N, N_{uf} \cap N^\circ\} \subseteq \mathbb{Z}$.

A lattice $M = \text{Hom}(N, \mathbb{Z})$ called the *character lattice* and sublattice $M^\circ = \text{Hom}(N^\circ, \mathbb{Z})$.

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- A set of positive integers $\{d_i\}_{i \in I}$ such that $\gcd(d_i) = 1$.

- A sublattice $N^o \subseteq N$ of finite index such that $\{N_{uf}, N^o\} \subseteq \mathbb{Z}$ and
  $\{N, N_{uf} \cap N^o\} \subseteq \mathbb{Z}$.

- A lattice $M = \text{Hom}(N, \mathbb{Z})$ called the *character lattice* and sublattice
  $M^o = \text{Hom}(N^o, \mathbb{Z})$. 
Scattering Diagrams for Generalized Cluster Algebras

To adapt the cluster scattering diagram construction of Gross, Hacking, Keel, and Kontsevich for the generalized setting, we begin by defining the 
generalized fixed data \( \Gamma \):

- The cocharacter lattice \( N \) with skew-symmetric bilinear form 
  \( \{ \cdot, \cdot \} : N \times N \to \mathbb{Q} \).
- A saturated sublattice \( N_{uf} \subseteq N \) called the \textit{unfrozen sublattice}.
- An index set \( I \) with \( |I| = \text{rank}(N) \) and subset \( I_{uf} \subseteq I \) such that 
  \( |I_{unf}| = \text{rank}(N_{uf}) \).
- A set of positive integers \( \{ d_i \}_{i \in I} \) such that \( \gcd(d_i) = 1 \).
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To construct cluster scattering diagrams, we must assume that the map

\[ p_1^* : N_{uf} \rightarrow M^o \]

\[ n \mapsto \{ n, \cdot \} \]

is injective.
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**Note:** This is not true for all choices of fixed data, but is true in the principal coefficient case.
Example: (In some sense, “generalized $G2$”)

Consider $\mathcal{A} = \left( x, y, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, (3, 1), ((1, a, a, 1), (1, 1)) \right)$. 
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Consider $\mathcal{A} = (x, y, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, (3, 1), ((1, a, a, 1), (1, 1)))$.

This generalized cluster algebra has fixed data:

- Skew-symmetric bilinear form given by the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
- Index sets $I = I_{uf} = \{1, 2\}$
- $d_1 = d_2 = 1$
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To explicitly describe the lattices $N, N^\circ, M,$ and $M^\circ$, we will need some choice of generalized torus seed.
A *generalized torus seed* is a collection \( s = \{(e_i, a_i)\}_{i \in I, s \in [r_i]} \) such that:

- \( \{e_i\}_{i \in I} \) is a basis for \( N \),
- \( \{d_i e_i\}_{i \in I} \) is a basis for \( N^\perp \),
- \( \{e^* i\}_{i \in I} \) is a basis for the dual lattice \( M \),
- \( \{f_i = d_i - 1\}_{i \in I} \) is a basis for \( M^\perp \),
- and each \( a_i = (a_{i_1}, s) \) is a tuple of scalars where \( a_{i_1}, s = 1 \).

The generalized torus seed data defines a new bilinear form \([ \cdot, \cdot ]_s: N \times N \rightarrow \mathbb{Q}\) given by \([e_i, e_j]_s = \epsilon_{ij} = \{e_i, e_j\} d_j\).

Note: Placing the exchange polynomial coefficients \( (a_i, s) \) in the seed data rather than fixed data leaves open the possibility of using this construction for non-reciprocal generalized cluster algebras.
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3. $\{d_i e_i\}_{i \in I}$ is a basis for $N^\circ$,
4. $\{e^*_i\}_{i \in I}$ is a basis for the dual lattice $M$,
5. $\{f_i = d_i^{-1} e^*_i\}_{i \in I}$ is a basis for $M^\circ$,
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Example: (In some sense, “generalized G2”)

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\begin{align*}
  f_{01} &= 1 + z^{(-1,0)} \\
  f_{02} &= 1 + az^{(0,1)} + az^{(0,2)} + z^{(0,3)} \\
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\[ C^+ = \bigoplus_i \mathbb{R}_{\geq 0} e_i \]

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Here, we choose the seed \( s = \{((1, 0), (1, a, a, 1)), ((0, 1), (1, 1))\} \)
In the language of cluster algebras:

The exchange matrix $B$ encodes the $\epsilon_{ij}$ (i.e., the skew-symmetric form and choice of $\{d_i\}_{i \in I}$).

The cluster variables are given by $x_i = z e_i$ (the $A$-variety).

The coefficients are given by $y_i = z f_i$ (the $X$-variety).

The index sets $I$ and $I_{uf}$ allow us to differentiate between frozen and unfrozen variables.

The classic mutation relations can be derived from the mutation of the $e_i$, $f_i$, and $\epsilon_{ij}$ via simple algebra.
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The classic mutation relations can be derived from the mutation of the $e_i, f_i,$ and $\epsilon_{ij}$ via simple algebra.
The generalized mutation relations for basis vectors $e_i, f_i$ and the $\epsilon_{i,j}$ are:

\[ e'_i := \begin{cases} e_i + r_k [\epsilon_{ik}] + e_k & i \neq k \\ -e_k & i = k \end{cases} \]

\[ f'_i := \begin{cases} -f_k + r_k \sum_{j \in I_{uf}} [-\epsilon_{kj}] + f_j & i = k \\ f_i & i \neq k \end{cases} \]

\[ \epsilon'_{ij} = \begin{cases} -\epsilon_{ij} & k = i \text{ or } k = j \\ \epsilon_{ij} & k \neq i, j \text{ and } \epsilon_{ik}\epsilon_{kj} \leq 0 \\ \epsilon_{ij} + r_k |\epsilon_{ik}|\epsilon_{kj} & k \neq i, j \text{ and } \epsilon_{ik}\epsilon_{kj} \geq 0 \end{cases} \]

And we add the relation $a'_{k,s} = a_{k,r_k-s}$.

**Note:** Seed mutation is only an involution up to isomorphism.
Example: (In some sense, “generalized G2”)
**Example:** (In some sense, “generalized $G_2$”) Recall that our running example has $d_1 = d_2 = 1$, $r_1 = 3$, and $r_2 = 1$. Let us choose the seed $s = \{(1, 0), (1, a, a, 1)\}, \{(0, 1), (1, 1)\}$. Then because $e_1 = e^*_1 = f_1 = (1, 0)$ and $e_2 = e^*_2 = f_2 = (0, 1)$, we have $N = N^\circ = M = M^\circ = \langle (1, 0), (0, 1) \rangle$.

Remark: Those familiar with cluster scattering diagrams might notice that these are not the same lattices as for an ordinary cluster algebra of type $G_2$ (for which $d_1 = 3$ and $d_2 = 1$) when choosing the ordinary torus seed $s = \{(1, 0), (0, 1)\}$. In that case, $N = M = \langle (1, 0), (0, 1) \rangle$, $N^\circ = \langle (3, 0), (0, 1) \rangle$, $M^\circ = \langle (1, 3), (0, 1) \rangle$. Elizabeth Kelley (joint work with Man-Wai Cheung and Gregg Musiker) Theta Basis for Generalized Cluster Algebras December 9, 2020 28 / 63
Example: (In some sense, “generalized $G2$”) 

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\begin{bmatrix}
1 & 0 \\
-1 & 0
\end{bmatrix}.
\]

So $\epsilon = [\epsilon_{ij}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. 

And therefore

\[\mu_2(s) = \{(1, 1), (0, -1)\}.\]
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- $e'_1 = (1, 0) + 1[1]_+(0, 1) = (1, 1)$
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- $f'_1 = (1, 0)$
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And therefore $\mu_2(s) = \{((1, 1), (1, a, a, 1)), ((0, -1), (1, 1))\}$. 
Generalized Cluster Varieties

Given a generalized torus seed $s$, we can define *generalized cluster varieties*. 

Associate some $s$ to vertex $v$ and associate the tori $X_s = T_M = \text{Spec} k[N]_A = T_N \circ = \text{Spec} k[M \circ]$. 

---

Elizabeth Kelley  (joint work with Man-Wai  Theta Basis for Generalized Cluster Algebras  December 9, 2020  30 / 63
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Let $T$ be an infinite $|I_{uf}|$-regular tree with structure shown below:
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Let $\mathcal{T}$ be an infinite $|I_{uf}|$-regular tree with structure shown below:

Associate some $s$ to vertex $v$ and associate the tori

$$\mathcal{X}_s = T_M = \text{Spec } k[N]$$
$$\mathcal{A}_s = T_{N^\circ} = \text{Spec } k[M^\circ]$$
We can then define birational maps between these tori.
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For $n \in \mathbb{N}$ and $m \in M^\circ$, we define birational maps $\mu_k : \mathcal{X}_s \to \mathcal{X}_{\mu_k(s)}$ and $\mu_k : \mathcal{A}_s \to \mathcal{A}_{\mu_k(s)}$ via the pull-back of functions

$$\mu_k^* z^n = z^n \left( 1 + a_{k,1} z^{e_k} + \cdots + a_{k,r_k-1} z^{(r_k-1)e_k} + z^{r_k e_k} \right)^{-[n,e_k]}$$

$$\mu_k^* z^m = z^m \left( 1 + a_{k,1} z^{v_k} + \cdots + a_{k,r_k-1} z^{(r_k-1)v_k} + z^{r_k v_k} \right)^{-\langle d_k e_k, m \rangle}$$

**Note:** These maps encode the exchange relations of the cluster variables and coefficients.
The generalized $\mathcal{X}$ and $\mathcal{A}$ cluster varieties are then defined as

$$
\mathcal{A} := \bigcup_{w \in \mathcal{I}} T_{N^o, s_w}, \quad \mathcal{X} := \bigcup_{w \in \mathcal{I}} T_{M, s_w}
$$

where the tori are glued according to the previous birational maps.
A scattering diagram is a collection of *walls* $\vartheta$ and automorphisms $f_0$.

Each wall is a codimension-1 cone (in rank 2, these are simply lines) and the automorphisms are formal power series in $z$. 
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\[
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\partial_1 & \quad \partial_2 & \quad \partial_3 & \quad \partial_4 & \quad \partial_5 & \quad \partial_6 \\
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f_{\partial_1} &= 1 + z^{(-1,0)} \\
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\end{align*}
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Some useful observations:

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- Each $g$-vector of the generalized cluster algebra appears as the support of a wall.
- Each chamber corresponds to a cluster seed.
The *initial scattering diagram* is defined as

\[ \mathcal{D}_{in,s} := \{ (e_i^\perp, 1 + a_{i,1}z^{v_i} + a_{i,2}z^{2v_i} + \cdots + a_{i,r_i-1}z^{(r_i-1)v_i} + z^{r_i v_i}) : i \in I_{uf} \} \]

where \( v_i = \{ e_i, \cdot \} \) for \( i \in I_{uf} \).
Given generalized fixed data $\Gamma$, the generalized fixed data for the cluster variety with principal coefficients, $\Gamma_{\text{prin}}$, is defined by:

The double of the lattice $N, \tilde{N} = N \oplus M$, with skew-symmetric bilinear form given by

$\{ (n_1, m_1), (n_2, m_2) \} = \{ n_1, n_2 \} + \langle n_1, m_2 \rangle - \langle n_2, m_1 \rangle$.

The unfrozen sublattice $\tilde{N}_{\text{uf}} = N_{\text{uf}} \oplus 0 \sim N_{\text{uf}}$.

The sublattice $\tilde{N}^\circ = N^\circ \oplus M$.

The index set $\tilde{I}$ is given by the disjoint union of two copies of $I$.

The integer collections $\tilde{d} = (d_i)$ and $\tilde{r} = (r_i)$ taken so that the $d_i$ and $r_i$ agree with $\Gamma$.

The unfrozen index set, $\tilde{I}_{\text{uf}}$, which is the original $I_{\text{uf}}$ thought of as a subset of the first copy of $I$.

The character lattice $\tilde{M} = \text{Hom}(\tilde{N}, \mathbb{Z}) = M \oplus N^\circ$ with sublattice $\tilde{M}^\circ = M^\circ \oplus N^\circ$. 

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Theta Basis for Generalized Cluster Algebras
Principal Coefficients

Given generalized fixed data $\Gamma$, the generalized fixed data for the cluster variety with principal coefficients, $\Gamma_{\text{prin}}$, is defined by:

- The *double* of the lattice $N$, $\tilde{N} := N \oplus M^\circ$, with skew-symmetric bilinear form given by

\[
\{(n_1, m_1), (n_2, m_2)\} = \{n_1, n_2\} + \langle n_1, m_2 \rangle - \langle n_2, m_1 \rangle.
\]
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Theta Basis for Generalized Cluster Algebras

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  \]

- The unfrozen sublattice $\tilde{N}_{uf} := N_{uf} \oplus 0 \cong N_{uf}$.
- The sublattice $\tilde{N}^\circ := N^\circ \oplus M$.
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- The integer collections $\tilde{d} = (d_i)$ and $\tilde{r} = (r_i)$ taken so that the $d_i$ and $r_i$ agree with $\Gamma$.
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Given a generalized torus seed $s$, the generalized torus seed with principal coefficients $s_{\text{prin}}$ is defined as $s_{\text{prin}} := \{(e_i, 0), (0, f_i)\}_{i \in I}$. 
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Let $\tilde{\nu}_i := (\nu_i, e_i) = (p_1^*(e_i), e_i)$. Then

$$\mathcal{O}^{A_{\text{prin}}}_{\text{in}, s} = \left\{ (e_i, 0)^\perp, 1 + a_{i, 1}z^{\tilde{\nu}_1} + \cdots + a_{i, r_i - 1}z^{(r_i - 1)\tilde{\nu}_i} + z^{r_i\tilde{\nu}_i} \right\}$$
**Example:** In our running example, we have

\[ \tilde{I} = \{1, 2\} \sqcup \{1, 2\} \]
\[ \tilde{d} = (1, 1, 1, 1) \]
\[ \tilde{r} = (3, 1, 3, 1) \]

and the lattices \( \tilde{N} = N \oplus M^\circ, \tilde{N}^\circ = N^\circ \oplus M, \tilde{M} = M \oplus N^\circ, \) and \( \tilde{M}^\circ = M^\circ \oplus N \) all have basis

\[ \langle (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1) \rangle \]

Our running choice of generalized torus seed becomes

\[ s_{\text{prin}} = \{((1, 0, 0, 0)^\perp, (1, a, a, 1)), ((0, 1, 0, 0)^\perp, (1, 1))\} \]
Principal Coefficients

Example: In our running example, $\mathcal{D}_{s\text{ prin}}$ is

\[
\begin{align*}
\tilde{f}_{\tilde{d}_1} &= 1 + z^{(-1,0,0,1)} \\
\tilde{f}_{\tilde{d}_2} &= 1 + az^{(0,1,1,0)} + az^{(0,2,2,0)} + z^{(0,3,3,0)} \\
\tilde{f}_{\tilde{d}_3} &= 1 + z^{(-1,3,3,1)} \\
\tilde{f}_{\tilde{d}_4} &= 1 + az^{(-1,2,2,1)} + az^{(-2,4,4,2)} + z^{(-3,6,6,3)} \\
\tilde{f}_{\tilde{d}_5} &= 1 + z^{(-2,3,3,2)} \\
\tilde{f}_{\tilde{d}_6} &= 1 + az^{(-1,1,1,1)} + az^{(-2,2,2,2)} + z^{(-3,3,3,3)}
\end{align*}
\]
**Principal Coefficients**

**Example:** In our running example, \( \mathcal{D}_{s_{\text{prin}}} \) is

\[
\begin{align*}
\tilde{f}_{\tilde{d}_1} &= 1 + z^{(-1,0,0,1)} \\
\tilde{f}_{\tilde{d}_2} &= 1 + az^{(0,1,1,0)} + az^{(0,2,2,0)} + z^{(0,3,3,0)} \\
\tilde{f}_{\tilde{d}_3} &= 1 + z^{(-1,3,3,1)} \\
\tilde{f}_{\tilde{d}_4} &= 1 + az^{(-1,2,2,1)} + az^{(-2,4,4,2)} + z^{(-3,6,6,3)} \\
\tilde{f}_{\tilde{d}_5} &= 1 + z^{(-2,3,3,2)} \\
\tilde{f}_{\tilde{d}_6} &= 1 + az^{(-1,1,1,1)} + az^{(-2,2,2,2)} + z^{(-3,3,3,3)}
\end{align*}
\]

**Note:** This diagram is actually four-dimensional, but is shown here as a projection onto \( M^o \).
As before, a choice of $s_{\text{prin}}$ defines tori

$$\mathcal{X}_{s_{\text{prin}}} := T_{\tilde{M}} = \text{Spec } \mathbb{k}[\tilde{N}],$$

$$\mathcal{A}_{s_{\text{prin}}} := T_{\tilde{N}^\circ} = \text{Spec } \mathbb{k}[\tilde{M}^\circ]$$

which are glued using the birational mutation maps to obtain the schemes $\mathcal{A}_{\text{prin}}$ and $\mathcal{X}_{\text{prin}}$. 
As before, a choice of $s_{\text{prin}}$ defines tori

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which are glued using the birational mutation maps to obtain the schemes $\mathcal{A}_{\text{prin}}$ and $\mathcal{X}_{\text{prin}}$.

The schemes $\mathcal{A}$ and $\mathcal{X}$ are given, respectively, by the fiber $\mathcal{A}_e$ and the quotient $\mathcal{A}_{\text{prin}}/T_{\mathcal{N}^\circ}$. 
This is reflected in the fact that we can obtain the $\mathcal{X}$ scattering diagram by taking the slice $\{m \in M^\circ : m = p_1^*(n)\}$ of the $A_{\text{prin}}$ scattering diagram.
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In our running example, this gives us:

\[
\begin{align*}
    f_{01} &= 1 + z^{(0,1)} \\
    f_{02} &= 1 + az^{(1,0)} + az^{(2,0)} + z^{(3,0)} \\
    f_{03} &= 1 + z^{(3,1)} \\
    f_{04} &= 1 + az^{(2,1)} + az^{(4,2)} + z^{(6,3)} \\
    f_{05} &= 1 + z^{(3,2)} \\
    f_{06} &= 1 + az^{(1,1)} + az^{(2,2)} + z^{(3,3)}
\end{align*}
\]
This is reflected in the fact that we can obtain the $\mathcal{X}$ scattering diagram by taking the slice $\{m \in M^\circ : m = p_1^*(n)\}$ of the $A_{\text{prin}}$ scattering diagram.

In our running example, this gives us:

\[
\begin{align*}
\varphi_1 &= 1 + z^{(0,1)} \\
\varphi_2 &= 1 + az^{(1,0)} + az^{(2,0)} + z^{(3,0)} \\
\varphi_3 &= 1 + z^{(3,1)} \\
\varphi_4 &= 1 + az^{(2,1)} + az^{(4,2)} + z^{(6,3)} \\
\varphi_5 &= 1 + z^{(3,2)} \\
\varphi_6 &= 1 + az^{(1,1)} + az^{(2,2)} + z^{(3,3)}
\end{align*}
\]

**Note:** Here, the $\mathcal{X}$ and $A$ scattering diagrams have the same dimension because $p_1^*$ is injective.
So far, we’ve only defined the initial scattering diagram.
So far, we’ve only defined the initial scattering diagram.

To explain how one could actually obtain these completed scattering diagrams in our previous examples, we need a little bit of additional framework.
Wall-crossing

Crossing a wall \((\partial, f_0)\) acts on monomials as \(z^m \mapsto z^m f_0^{\langle n_0, m \rangle}\), where \(n_0\) is the primitive vector normal to \(\partial\) that opposes the direction of travel. It acts on a polynomial by acting on each individual monomial.
Wall-crossing

Crossing a wall \((\partial, f_0)\) acts on monomials as \(z^m \mapsto z^m f_0^{\langle n_0, m \rangle}\), where \(n_0\) is the primitive vector normal to \(\partial\) that opposes the direction of travel.

It acts on a polynomial by acting on each individual monomial.

\[
\begin{align*}
z^{(2,-3)} &\mapsto z^{(2,-3)} \left( 1 + az^{(-1,1)} + az^{(-2,2)} + z^{(-3,3)} \right)^{(2,3),(-1,-1)} \\
&= z^{(2,-3)} \left( 1 + az^{(-1,1)} + az^{(-2,2)} + z^{(-3,3)} \right)
\end{align*}
\]
Path-ordered Products

Composing multiple wall-crossings along a path $\gamma$ gives us the *path-ordered product* $p_\gamma$. 

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Composing multiple wall-crossings along a path $\gamma$ gives us the path-ordered product $p_\gamma$.

$$z^{(2,-3)} \mapsto z^{(2,-3)} \left( 1 + az^{(-1,1)} + az^{(-2,2)} + z^{(-3,3)} \right)^{(2,3),(-1,-1)}$$

$$= z^{(2,-3)} \left( 1 + az^{(-1,1)} + az^{(-2,2)} + z^{(-3,3)} \right)$$

$$\mapsto z^{(2,-3)} \left( 1 + z^{(-1,0)} \right)^3.$$
A scattering diagram $\mathcal{D}$ is called \textit{consistent} if $p_\gamma = 1$ for any closed loop $\gamma$.

We build $\mathcal{D}$ from the initial scattering diagram $\mathcal{D}_{\text{in}}$ by adding any walls necessary to satisfy this condition.
A scattering diagram $\mathcal{D}$ is called \textit{consistent} if $p_\gamma = 1$ for any closed loop $\gamma$.

We build $\mathcal{D}$ from the initial scattering diagram $\mathcal{D}_{\text{in}}$ by adding any walls necessary to satisfy this condition.

In our running example, $\mathcal{D}_{\text{in}}$ consists of the vertical and horizontal walls and we could have used consistency to add the remaining walls.
Two scattering diagrams $\mathcal{D}$ and $\mathcal{D}'$ are *equivalent* if $p_{\gamma,\mathcal{D}} = p_{\gamma,\mathcal{D}'}$ for all paths $\gamma$ for which both path-ordered products are defined.

The following holds in the generalized setting:

**Theorem:** (GHKK, 2018) Given fixed data $\Gamma$ and seed $s$, there is a consistent scattering diagram $\mathcal{D}_s$ which contains $\mathcal{D}_{in,s}$ such that $\mathcal{D}_s \setminus \mathcal{D}_{in,s}$ consists only of outgoing walls. The scattering diagram $\mathcal{D}_s$ is unique up to equivalence.
Mutation Invariance:

Because mutation equivalent seeds $s$ and $s'$ generate the same generalized cluster algebra, we should expect $\mathcal{D}_s$ and $\mathcal{D}_{s'}$ to be equivalent.
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Because mutation equivalent seeds $s$ and $s'$ generate the same generalized cluster algebra, we should expect $\mathcal{D}_s$ and $\mathcal{D}_{s'}$ to be equivalent.

Let

$$\mathcal{H}_{k,+} := \{ m \in M_\mathbb{R} : \langle e_k, m \rangle \geq 0 \},$$
$$\mathcal{H}_{k,-} := \{ m \in M_\mathbb{R} : \langle e_k, m \rangle \leq 0 \}.$$
Because mutation equivalent seeds $\mathbf{s}$ and $\mathbf{s}'$ generate the same generalized cluster algebra, we should expect $\mathcal{D}_\mathbf{s}$ and $\mathcal{D}_{\mathbf{s}'}$ to be equivalent.

Let

\[
\begin{align*}
\mathcal{H}_{k,+} & := \{ m \in \mathcal{M}_\mathbb{R} : \langle \mathbf{e}_k, m \rangle \geq 0 \}, \\
\mathcal{H}_{k,-} & := \{ m \in \mathcal{M}_\mathbb{R} : \langle \mathbf{e}_k, m \rangle \leq 0 \}.
\end{align*}
\]

The birational maps $\mu_k : \mathcal{A}_\mathbf{s} \rightarrow \mathcal{A}_{\mu_k(\mathbf{s})}$ and $\mu_k : \mathcal{X}_\mathbf{s} \rightarrow \mathcal{X}_{\mu_k(\mathbf{s})}$ tropicalize to the piecewise linear map

\[
T_k(m) := \begin{cases} 
m + r_k v_k \langle d_k \mathbf{e}_k, m \rangle & m \in \mathcal{H}_{k,+} \\
m & m \in \mathcal{H}_{k,-}
\end{cases}
\]
We compute $T_k(\mathcal{D})$ by:

1. Replacing the wall $(e_k^\perp, 1 + a_{k,1}z^{v_k} + \cdots + a_{k,r_k-1}z^{(r_k-1)v_k} + z^{r_kv_k})$ with $(e_k^\perp, 1 + a_{k,1}z^{-v_k} + \cdots + a_{k,r_k-1}z^{-(r_k-1)v_k} + z^{-r_kv_k})$.

2. Applying $T_k$ to the support and wall-crossing automorphism of each remaining wall.
Applying $T_1$ to our running example, we get:

\[\begin{array}{c}
\begin{array}{c}
\varnothing_2 \\
\varnothing_3 \\
\varnothing_4 \\
\varnothing_5 \\
\varnothing_6 \\
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\varnothing'_{1'} \\
\varnothing'_{2'} \\
\varnothing'_{3'} \\
\varnothing'_{4'} \\
\varnothing'_{5'} \\
\varnothing'_{6'} \\
\end{array}
\end{array}\]

Note: $T_1$ is only an involution up to equivalence of diagrams.

Elizabeth Kelley (joint work with Man-Wai Cheung and Gregg Musiker)

Theta Basis for Generalized Cluster Algebras

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Applying $T_1$ to our running example, we get:

Note: $T_k$ is only an involution up to equivalence of diagrams.
Mutation Invariance:

Theorem (Cheung-K.-Musiker): (c.f. Theorem 1.24 of GHKK) If the injectivity assumption holds, then $T_k(\mathcal{D}_s)$ is a consistent scattering diagram for $N^{\mu_k}_+(s)$. Moreover, the diagrams $\mathcal{D}_{\mu_k}(s)$ and $T_k(\mathcal{D}_s)$ are equivalent.

The proof of this theorem is quite lengthy, but one key point is that it relies on the reciprocity condition $a_{k,i} = a_{k,r_k-i}$. 
Building $\mathcal{A}$ from the scattering diagram

For a given scattering diagram $\mathcal{D}_s$, we build a scheme $\mathcal{A}_{\text{scat}}$ by associating tori to each chamber and gluing along the birational mutation maps. We need to show that $\mathcal{A}_{\text{scat}}$ is isomorphic to $\mathcal{A}_s$. A key step in doing so is checking the commutativity of the following diagram for mutation equivalent seeds $s$ and $s'$.

\[
\begin{array}{c}
T \circ \sigma,
\end{array}
\begin{array}{c}
\tilde{\sigma}
\end{array}
\begin{array}{c}
T \circ \sigma,
\end{array}
\begin{array}{c}
\tilde{\sigma}
\end{array}
\begin{array}{c}
Tv'\sigma
\end{array}
\begin{array}{c}
p\sigma,\tilde{\sigma}
\end{array}
\begin{array}{c}
p\sigma',\tilde{\sigma}'
\end{array}
\begin{array}{c}
Tv'\tilde{\sigma}
\end{array}
\]

where $\sigma$ and $\tilde{\sigma}$ are chambers in some $\mathcal{D}_s$ and $\sigma'$ and $\tilde{\sigma}'$ are the corresponding chambers in $\mathcal{D}_s'$. This again requires the reciprocity condition.
Building $\mathcal{A}$ from the scattering diagram

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A key step in doing so is checking the commutativity of the following diagram for mutation equivalent seeds $s$ and $s'$.

$$
\begin{array}{ccc}
TN^\circ,\sigma & \xrightarrow{T_{v'},\sigma} & TN^\circ,\sigma' \\
\downarrow p_{\sigma,\bar{\sigma}} & & \downarrow p_{\sigma',\bar{\sigma}'} \\
TN^\circ,\tilde{\sigma} & \xrightarrow{T_{v'},\tilde{\sigma}} & TN^\circ,\tilde{\sigma}'
\end{array}
$$

where $\sigma$ and $\tilde{\sigma}$ are chambers in some $\mathcal{D}_s$ and $\sigma'$ and $\tilde{\sigma}'$ are the corresponding chambers in $\mathcal{D}_{s'}$. 

This again requires the reciprocity condition.

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Building $\mathcal{A}$ from the scattering diagram

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We need show that $\mathcal{A}_{\text{scat}}$ is isomorphic to $\mathcal{A}$.

A key step in doing so is checking the commutativity of the following diagram for mutation equivalent seeds $s$ and $s'$.

\[
\begin{array}{ccc}
T_{N^\circ,\sigma} & \xrightarrow{T_{v',\sigma}} & T_{N^\circ,\sigma'} \\
\downarrow{p_{\sigma,\tilde{\sigma}}} & & \downarrow{p_{\sigma',\tilde{\sigma}'}} \\
T_{N^\circ,\tilde{\sigma}} & \xrightarrow{T_{v',\tilde{\sigma}}} & T_{N^\circ,\tilde{\sigma}'}
\end{array}
\]

where $\sigma$ and $\tilde{\sigma}$ are chambers in some $\mathcal{D}_s$ and $\sigma'$ and $\tilde{\sigma}'$ are the corresponding chambers in $\mathcal{D}_{s'}$.

This again requires the reciprocity condition.
Theorem (Cheung-K.-Musiker): (c.f. Theorem 4.4 of GHKK) Let $s$ be a generalized torus seed, $v$ be the root of $\mathcal{Z}_s$ and $v'$ be any other vertex. Let $\psi^*: M^\circ_{v'} \to M^\circ_v$ be the linear map $\mu_{v,v'}^{T_{v,v'}}|_{C^+_v \in s}$ and $\psi_{v,v'}: T_{N^\circ, v'} \to T_{N^\circ, v'}$ be the map between the associated tori. Then the collection $\{\psi_{v,v'}\}_{v'}$ glue to give an isomorphism

$$A_s := \bigcup_{v'} T_{N^\circ} \to A_{\text{scat}, s} := \bigcup_{v'} T_{N^\circ, v'}$$

and the diagram

$$\begin{array}{ccc}
A_s & \longrightarrow & A_{\text{scat}, s} \\
\downarrow & & \downarrow \\
A_{s_{v'}} & \longrightarrow & A_{\text{scat}, s_{v'}}
\end{array}$$

commutes.
Building $\mathcal{A}$ from the scattering diagram

The upshot:
We can identify the rings of regular functions on $\mathcal{A}_{\text{scat,} s}$ and $\mathcal{A}_s$. 
Building $A$ from the scattering diagram

The upshot:
We can identify the rings of regular functions on $A_{\text{scat}}$, $s$ and $A_s$.

This is a key component of showing that the theta functions form a basis.
Theta functions can be defined in terms of path-ordered products as

\[ \vartheta_m = p_\gamma(z^m) \]

where \( \gamma \) is a path to the positive chamber.
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\[ \vartheta_m = p_\gamma(z^m) \]

where \( \gamma \) is a path to the positive chamber.

These path-ordered products are easy to compute in areas where the walls aren’t dense. In many scattering diagrams, though, there are dense regions:

When \( bc \geq 5 \), every wall with rational slope appears inside the shaded cone.

(“The badlands”)
This motivates *broken lines*, which give another way to define $\vartheta_m$. 

Roughly, a broken line is a collection of all piecewise linear paths which begin with slope $m_0$, "scatter" off the walls in particular ways dictated by the wall-crossing automorphisms, and end at a particular point $Q$. 

Let $m_0 = (0, -1)$ and $Q$ be below the diagonal: 

$z((-1, 2), z((-1, -1), z(0, -1), az((-1, 1), z((-1, -1), z(0, -1), z((-1, -1), z(0, -1)$. 

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This motivates *broken lines*, which give another way to define $\vartheta_m$.

Roughly, a broken line is a collection of all piecewise linear paths which begin with slope $-m_0$, “scatter” off the walls in particular ways dictated by the wall-crossing automorphisms, and end at a particular point $Q$. 

Let 

\[
\begin{align*}
&z(-1,2) \quad z(-1,-1) \\
&z(0,-1) \quad z(-1,0) \quad z(-1,-1) \quad z(0,0) \quad z(0,-1)
\end{align*}
\]
This motivates *broken lines*, which give another way to define $\vartheta_m$.

Roughly, a broken line is a collection of all piecewise linear paths which begin with slope $-m_0$, “scatter” off the walls in particular ways dictated by the wall-crossing automorphisms, and end at a particular point $Q$.

Let $m_0 = (0, -1)$ and $Q$ be below the diagonal:
In terms of broken lines, the *theta basis* is defined as

$$\vartheta_{Q,m_0} := \sum_{\gamma} \text{Mono}(\gamma)$$

where the summation ranges over all broken lines $\gamma$ with initial slope $-m_0$ and endpoint $Q$ and $\text{Mono}(\gamma)$ denotes the monomial attached to the final domain of linearity.
\[ Q, (0, -1) = z^{(0, -1)} + z^{-1, -1} + az^{-1, 0} + az^{-1, 1} + z^{-1, 2} \]
\[ \psi_{Q,(0,-1)} = z^{(0,-1)} + z^{(-1,-1)} + az^{(-1,0)} + az^{(-1,1)} + z^{(-1,2)} \]

\[ = \frac{1 + x_1 + ax_2 + ax_2^2 + x_2^3}{x_1 x_2} \]
Now that we have defined theta functions, we can establish some necessary intermediate results for proving that the theta functions form a basis.
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$$\vartheta^{\mu_k(s)}_{T_k(Q), T_k(m_0)} = T_{k, \pm} \left( \vartheta^{s}_{Q, m_0} \right)$$

for $Q \in \mathcal{H}_{k, \pm}$ where $T_{k, \pm}$ acts linearly on the exponents in $\vartheta^{s}_{Q, m_0}$.
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**Note:** The proof of this proposition also requires the reciprocity condition.
Theta basis

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Lemma (Cheung-K.-Musiker): (c.f. Definition-Lemma 6.2 and Proposition 6.3 of GHKK)

Let $p_1, p_2$, and $q$ be points in $\tilde{M}_s^\circ$ and $z$ be a generic point in $\tilde{M}_{\mathbb{R},s}^\circ$. There are at most finitely many pairs of broken lines $\gamma_1, \gamma_2$ such that $\gamma_i$ has initial slope $p_i$, both broken lines have endpoint $z$, and $F(\gamma_1) + F(\gamma_2) = q$. Let $a_z(p_1, p_2, q) := \sum_{(\gamma_1, \gamma_2)} c(\gamma_1)c(\gamma_2)$ for pairs $\gamma_1, \gamma_2$ such that $I(\gamma_i) = p_i, b(\gamma_i) = z$, and $F(\gamma_1) + F(\gamma_2) = q$. Then

$$\vartheta_{p_1} \cdot \vartheta_{p_2} = \sum_{q \in \tilde{M}_s^\circ} \alpha_z(q)(p_1, p_2, q)\vartheta_q$$

for $z(q)$ sufficiently close to $q$. When $z$ is sufficiently close to $q$, $a_z(p_1, p_2, q)$ is independent of the choice of $z$ and we can simply write $\alpha(p_1, p_2, q) := a_z(p_1, p_2, q)$. 

Elizabeth Kelley (joint work with Man-Wai Cheung and Gregg Musiker)

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All of which come together to allow us to prove:

**Theorem (Cheung-K.-Musiker):** Given a set of generalized fixed data $\Gamma$, the collection $\{\vartheta_{Q,m}\}_{m \in \tilde{M}^\circ}$ forms a basis for the associated reciprocal generalized cluster algebra.
References


