SOLUTIONS – EXAM I

> Be sure to answer a total of FIVE QUESTIONS: Complete all questions 1 through 4, and then either 5(a) or 5(b). Answer the questions in the space provided on the question sheets. If you need extra space, write on the other side of the page, in this case please clearly indicate that your work is continued on the other side.

1. (20 points) Determine if the following pairs of statements are equivalent. In each part, provide proof if the statements are equivalent, or if they are not, provide a case where they have different truth values.

(a) $(P \wedge Q) \vee R$ versus $P \wedge (Q \vee R)$.

NOT EQUIVALENT

Have different truth values if
P is false while Q is true & R is true
if
P is false & Q is false, while R is true

P	Q	R	$(P\wedge Q)$	$(Q\vee R)$	$(P\wedge Q)\vee R$	$P\wedge(Q\vee R)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	T	T
T	F	F	F	F	F	F
F	T	T	F	T	T	F
F	T	F	F	T	F	F
F	F	T	F	T	T	F
F	F	F	F	F	F	F

(b) $(P \wedge Q) \implies R$ versus $(P \implies R) \vee (Q \implies R)$.

$$(P\wedge Q)\Rightarrow R \cong R\vee[\neg(P\wedge Q)]$$
$$\cong R\vee[\neg P\vee\neg Q]$$
$$\cong (R\vee R)\vee[\neg P\vee\neg Q]$$
$$\cong (R\vee\neg P)\vee(R\vee\neg Q)$$
$$\cong (P\Rightarrow R)\vee(Q\Rightarrow R)$$

Equivalent ✓

Alternatively,

$$(P\Rightarrow R)\vee(Q\Rightarrow R) \cong (R\vee\neg P)\vee(R\vee\neg Q)$$
$$\cong (R\vee R)\vee(\neg P\vee\neg Q)$$
$$\Updownarrow$$
$$\cong R\vee[\neg(P\wedge Q)]$$
$$\cong (P\wedge Q)\Rightarrow R$$

Equivalent ✓

OR ELSE

P	Q	R	$(P\wedge Q)$	$\neg P$	$\neg Q$	$\neg(P\wedge Q)$	$R\vee\neg P$ $P\Rightarrow R$	$R\vee\neg Q$ $Q\Rightarrow R$	$R\vee(\neg(P\vee Q))$ $(P\wedge Q)\Rightarrow R$	$(P\Rightarrow R)\vee(Q\Rightarrow R)$
T	T	T	T	F	F	F	T	T	T	T
T	T	F	T	F	F	F	F	F	F	F
T	F	T	F	F	T	T	T	T	T	T
T	F	F	F	F	T	T	F	T	T	T
F	T	T	F	T	F	T	T	T	T	T
F	T	F	F	T	F	T	T	F	T	T
F	F	T	F	T	T	T	T	T	T	T
F	F	F	F	T	T	T	T	T	T	T

Truth values all match.

2. (20 points) Quantify the following statements using the mathematical expression

$$P(x,y):\ x < y.$$

Your responses should contain no use of the symbols "<" or ">."

(a) Given any two real numbers, a and b with $a < b$, their average, $\frac{a+b}{2}$, is greater than a but less than b.

Acceptable answer: $\boxed{\forall a \in \mathbb{R}, \forall b \in \mathbb{R}\left[P(a,b) \Rightarrow \left[P\left(a, \frac{a+b}{2}\right) \& P\left(\frac{a+b}{2}, b\right)\right]\right]}$

Comments: Notice that "Given any" translates to "$\forall$" & "but" translates to "&". Anytime you have a statement with the form: "Given objects with conditions, conclusions" there is always an implication present. The statement could be rephrased:

Given two real numbers a & b, if $P(a,b)$ then $P\left(a,\frac{a+b}{2}\right) \& P\left(\frac{a+b}{2},b\right)$.

(b) Given any positive number less than 1, its square is less than itself.

Acceptable answer: $\boxed{\forall x \in \mathbb{R}\ [[P(0,x) \& P(x,1)] \Rightarrow P(x^2,x)]}$

or

$\boxed{\forall x \in (0,1)\ P(x^2,x)}$.

Comments: In the second answer the conditions on x are built into the set we're quantifying over.

(c) The negation of (b). The negation of (b) is the simplest statement that would make (b) false. For (b) to be false we would only need that: there's some positive number less than 1 for which its square is not less than itself.

Thus

Acceptable answers:

$\boxed{\exists x \in \mathbb{R}\ [[P(0,x) \& P(x,1)] \& \neg P(x^2,x)]}$

or

$\boxed{\exists x \in (0,1)\ \neg P(x^2,x)}$

3. (20 points) Consider the following statement:

P Q

(∗) If A and B are subsets of $\mathbb{R}$ such that $A \cap B$ is infinite, then both A and B are infinite.

(a) Write the contrapositive of (∗).

P: $A \cap B$ IS INFINITE Q: BOTH A AND B ARE INFINITE

$\sim Q$: ONE OF A AND B ARE FINITE

$\sim P$: $A \cap B$ FINTE. THE CONTRAPOSITIVE IS

$\sim Q \Rightarrow \sim P$: IF A, B SUBSETS OF $\mathbb{R}$ AND ONE OF A AND B ARE FINITE, THEN $A \cap B$ IS FINITE

(b) Prove that (∗) is true.

WE PROVE THE CONTRAPOSITIVE IN (a)

ASSUME A IS FINITE. THEN WE KNOW $A \cap B \subset A$

WE NOW NEED TO PROVE: $C \subset D$ AND D FINITE $\Rightarrow$ C FINITE

FOR ANY SETS C AND D

$D = \{d_1, \ldots, d_n\}$. $C \subset D \Rightarrow C$ CONTAINS k MEMBERS OF D, WHERE $0 \leq k \leq n$. SO $C = \{a_{j_1}, a_{j_2}, \ldots, a_{j_k}\}$ AND NO 2 ARE EQUAL. HENCE C FINITE & $\#C = k$

(c) Write the converse of (∗).

THE CONVERSE IS $Q \Rightarrow P$

IF A, B SUBSETS OF $\mathbb{R}$ SUCH THAT A AND B ARE INFINITE, THEN $A \cap B$ IS INFINITE

(d) Is the converse of (∗) true or false? Justify your answer with either a proof or a counterexample.

THE CONVERSE IS FALSE

COUNTER EXAMPLE: A = EVENS $= \{2, 4, 6 \ldots\} \subset \mathbb{N}$

B = ODDS $= \{1, 3, 5 \ldots\} \subset \mathbb{N}$

BOTH A AND B ARE INFINITE, BUT $A \cap B = \emptyset$

Solution to problem 4 [NBS]

$P(n): \quad 5^n + 6^n < 7^n \quad (n \in \mathbb{N})$

(a) Find the smallest $n_0 \in \mathbb{N}$ such that $P(n_0)$ is true.

Solution. $n_0 = 1$ fails: $5 + 6 = 11 > 7$.

$n_0 = 2$ fails: $5^2 + 6^2 = 25 + 36 > 49 = 7^2$.

$n_0 = 3$ succeeds: $5^3 + 6^3 = 125 + 216 < 343 = 7^3$.

Thus $n_0 = 3$ is the smallest.

(b) Prove by induction that $P(n)$ is true for all $n \in \mathbb{N}$ such that $n \geq n_0$ $(= 3)$.

Solution. (Basis step) $P(n_0) \equiv P(3)$ is true, as shown in part (a).

(Inductive step) Assume that $P(n)$ is true for some $n \geq n_0$, that is, that $5^n + 6^n < 7^n$.

We will prove that then $P(n+1)$ is true: $5^{n+1} + 6^{n+1} < 7^{n+1}$.

Start with the right-hand side:

$$
\begin{aligned}
7^{n+1} = 7 \cdot 7^n &> 7(5^n + 6^n) && \text{(by the inductive hypothesis)} \\
&= 7 \cdot 5^n + 7 \cdot 6^n \\
&> 5 \cdot 5^n + 6 \cdot 6^n && \text{(since } 7 > 5 \text{ and } 7 > 6\text{)} \\
&= 5^{n+1} + 6^{n+1}, && \text{as desired.}
\end{aligned}
$$

$\square$

MATH3283W EXAM1 SOLUTIONS

5.

(a) Consider the set

$$A = \{\frac{n-1}{n+1} | n \in \mathbb{N}\}$$

i Show that A is bounded above abd bounded below.

$\frac{n-1}{n+1} = \frac{n+1-2}{n+1} = 1 - \frac{2}{n+1} < 1$ for any $n \in \mathbb{N}$. So A is bounded above by 1.
On the other hand,

$$n-1 \geq 0, n+1 > 0 \Rightarrow \frac{n-1}{n+1} \geq 0$$

So A is bounded below by 0.

ii Find $L = \inf A$ and $M = \sup A$ (no proof is necessary in this part). Determine whether or not $L \in A$. Determine whether or not $M \in A$.

Since $1 - \frac{2}{n+1}$ is an increasing sequence, the infimum is the first term and the supremum is the limit of the sequence.

$$L = \inf A = \frac{1-1}{1+1} = 0, 0 \in A$$

$$M = \sup A = \lim_{n \to \infty} \frac{n-1}{n+1} = 1, 1 \notin A$$

iii Prove the value $M = \sup A$ you found in (ii) is in fact the least upper bound of A.

We know that 1 is an upper bound of A from (i). We also have to show:
For any $\varepsilon >, \exists x \in A$ such that $1 - \varepsilon < x \leq 1$

i.e. we want to find $n \in \mathbb{N}$ such that

$$\begin{aligned} & 1-\varepsilon < \frac{n-1}{n+1} = 1 - \frac{2}{n+1} \\ \Leftrightarrow \quad & -\varepsilon < -\frac{2}{n+1} \\ \Leftrightarrow \quad & \varepsilon > \frac{2}{n+1} \\ \Leftrightarrow \quad & n+1 > \frac{2}{\varepsilon} \end{aligned}$$

By Archimedean Property, we can find $n \in \mathbb{N}$ such that $n > \frac{2}{\varepsilon}$. Then

$$n+1 > n > \frac{2}{\varepsilon} \Leftrightarrow 1-\varepsilon < \frac{n-1}{n+1}$$

So 1 is the least upper bound of A.

(b) Determine if the following sets are bounnded above or below. In each case, if the set is bounded above, find the supremum; if the set is bounded below, find the infimum.

i $\{3+\frac{1}{2}, -2+\frac{1}{2}, 3+\frac{1}{4}, -2+\frac{1}{4}, 3+\frac{1}{8}, -2+\frac{1}{8}, \cdots\}$

The set is formed by two sequences:

$$A = \{3+\frac{1}{2}, 3+\frac{1}{4}, \cdots, 3+\frac{1}{2^n}, \cdots\}$$

and

$$B = \{-2+\frac{1}{2}, -2+\frac{1}{4}, \cdots, -2+\frac{1}{2^n}, \cdots\}$$

A is a decreasing sequences with limit 3. So $\sup A = 3\frac{1}{2}$ (first term) and $\inf A = 3$. Similarly, $\sup B = -1\frac{1}{2}$ (first term) and $\inf B = -2$ (the limit). Hence the set has supremum $3\frac{1}{2}$ and infimum -2.

ii $\{x \in \mathbb{R} | x > 0 \text{ and } x^2 - 4x + 3 > 0\}$

$$x^2 - 4x + 3 = (x-1)(x-3) = 0 \Rightarrow x = 1, 3$$

It is easy to show that $x^2 - 4x + 3 > 0$ when $x < 1$ or $x > 3$. Combine with the condition $x > 0$, we know that the set is $(0,1) \cup (3, \infty)$. So it is not bounded above but bounded below with $\inf = 0$.

iii $\{x \in \mathbb{R} | x^3 - x < 0\}$

$$x^3 - x = x(x-1)(x+1) = 0 \Rightarrow x = -1, 0, 1$$

It is easy to show that $x^3 - x < 0$ when $x < -1$ or $0 < x < 1$. So the set is $(-\infty, -1) \cup (0, 1)$. It is not bounded below but bounded above with sup $= 1$.

iv $\{1-.3, 2-.33, 3-.333, 4-.3333, 5-.33333, \cdots\} \cup \{\frac{1}{\sqrt{n}} | n \in \mathbb{N}\}$

The first set is an increasing sequence. It goes to ∞ when $n \to \infty$. The second set is a decreasing sequence with limit 0 by Archimedean property (or its corollary). So the set is not bounded above but bounded below with inf $= 0$.