**PART A**

1a) The roots of \( x^2 + 2x = 0 \) are \( x = 0 \) & \( x = -2 \). \( \forall x \in (-2, 0) \), \( x^2 + 2x < 0 \). For every \( x < -2 \), \( x^2 + 2x > 0 \). Thus

\[ \{ x \in \mathbb{R} \mid x < 0 \& \ x^2 + 2x > 0 \} = (-\infty, -2). \]

We see that the set is not bounded below, but is bounded above with supremum \(-2\).

1b) Notice that \( \frac{1}{2}, -\frac{1}{2}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{4}, -\frac{3}{4}, \ldots \)

\[ = \left\{ \frac{m}{n+1} \mid n \in \mathbb{N} \right\} \cup \left\{ \frac{n}{m+1} \mid n \in \mathbb{N} \right\}. \]

This set is bounded above & below.

The supremum is 1 & the infimum is -1.

To prove it let \( x > 0 \). Then by theorem 2.14 (1think)

\[ \exists n \in \mathbb{N} \text{ s.t. } 0 < \frac{1}{n} < x. \]

It follows that \( 1 - x < 1 - \frac{1}{n} = \frac{n-1}{n} \), & \( -1 + x > -1 + \frac{1}{n} = -\frac{n-1}{n} > 0 \). Thus given any number \( y \) less than 1 there is an element of the set greater than \( y \) & likewise for the infimum.

1c) We have \( \ldots, 1.9, 1.99, 1.999, \ldots \)

\[ = \left\{ \frac{m}{10^n} \mid n \in \mathbb{N} \right\} = \left\{ \frac{1}{10^n} \mid n \in \mathbb{N} \right\}. \]

This set is bounded with infimum 0 & supremum 1.
2. a) Since \( A = \{ \sin(x) \mid x \in \mathbb{R} \} = [-1, 1] \), it follows that
its bounded.

b) Notice \( \frac{\pi}{4} < \frac{\pi}{2} < \frac{3\pi}{2} < \frac{7\pi}{4} \).

Thus \( \sin(\frac{\pi}{2}) = 1 \in A \) & \( \sin(\frac{3\pi}{2}) = -1 \in A \).

Since \( A \subseteq [-1, 1] \) we have \( \sup(A) = 1 \) & \( \inf(A) = -1 \).

c) By part b) both \( \sup(A) \) & \( \inf(A) \) are elements of \( A \).

3a) Pf: Let \( R \) be an upper bound for \( A \),

Then \( x \leq R \forall x \in A \) & thus \( -R \leq -x \forall x \in A \).

Since \( -x \in A \) \( \forall x \in A \) we find \( -R \leq y \forall y \in A \).

That is, \( -A \) is bounded below.

b) Pf: By definition, \( w \) is an upper bound of \( A \).

Thus by 3a) \( -w \) is a lower bound for \( -A \).

Let \( r \) be any lower bound for \( -A \). Then

\( r \) is an upper bound for \( A \) since \( -R \leq -x \forall x \in A \)

& so \( r \geq x \forall x \in A \). Since \( w = \sup A \), \( r \geq w \),

& thus \( -R \leq -w \). That is, given a lower bound \( r \)
for \( -A \), \( -R \leq -w \), \( \therefore -w = \inf(-A) \).
4. We're given that $B \subseteq A$. To avoid messiness let $\inf(A) = -\infty$ when $A$ isn't bounded below.

Claim: $\inf(A) \leq \inf(B)$.

\textbf{Pf of claim: By definition, $B \subseteq A \iff \forall x \in B, x \in A$.}

Let $r = \inf(A)$ (r may be $-\infty$). Since $r \leq x$ for all $x \in A$ it follows that $r \leq x$ for all $x \in B$. That is, $r$ is a lower bound for $B$. By definition, if $r' = \inf(B)$, $r' \geq s$ for any lower bound $s$ of $B$, including $r$. i.e. $r \leq r'$. That is $\inf(A) \leq \inf(B)$.

\underline{PART B:}

5. First observe: $4^k > k^4 \iff (2^2)^k > (k^2)^2 \iff (2^k)^2 > (k^2)^2 \iff 2^k > k^2$. That is $\forall k \in \mathbb{N} [4^k > k^4] \iff [2^k > k^2]$.

Let's investigate $2^k$ & $k^2$ for some $k$'s:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$2^k$</th>
<th>$k^2$</th>
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<tbody>
<tr>
<td>1</td>
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<tr>
<td>5</td>
<td>32</td>
<td>25</td>
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Claim: $2^k > k^2 \forall k \geq 5$.

\textbf{Pf of claim: (By induction)}

\textbf{Base case:} Let $k = 5$. Then $2^5 = 32 > 25 = 5^2$. 
Inductive step: Suppose for some $k \geq 5$, $2^k > k^2$.

Then we have

$$2^{k+1} = 2 \cdot 2^k > 2k^2.$$  

So if we can prove that $2k^2 \geq (k+1)^2$ for $k \geq 5$ then it will follow that $2^{k+1} > 2k^2 \geq (k+1)^2$ & the proof will be complete.

Well

$$2k^2 \geq (k+1)^2.$$  

$$\iff 2k^2 \geq k^2 + 2k + 1$$  

$$\iff k^2 - 2k - 1 \geq 0$$

The roots of $x^2 - 2x - 1 = 0$ are $x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)} }{2} = 1 \pm \sqrt{2}$.  

If $x > 1 + \sqrt{2}$ we have $x^2 - 2x - 1 > 0$.

Since $k \geq 5$ we have $k^2 - 2k - 1 > 0$ & so $2k^2 \geq (k+1)^2$, & $2^{k+1} > (k+1)^2$.

We have proven that $\forall k \geq 5$, $2^k > k^2$ & by our initial observations we conclude $\forall k \geq 5$, $4^k > k^4$, as desired.
(a) The \((2k-1)^{st}\) element of \(A\) can be written
\[
2 + \sum_{j=1}^{k} \frac{1}{10^j} = 2 + \frac{k}{10} \sum_{j=1}^{K} \left(\frac{1}{10}\right)^j
\]

& the \((2k)^{th}\) element of \(A\) has the form
\[
-2 - \sum_{j=1}^{k} \frac{3}{10^j} = -2 - \frac{3k}{10} \sum_{j=1}^{K} \left(\frac{1}{10}\right)^j
\]

(b) It is clear that every odd element is greater than 2 & every even element is less than -2. Hence an upper bound of the odd elements will be an upper bound for \(A\) & likewise a lower bound of the even elements gives us a lower bound for \(A^e\).

From class we know
\[
\sum_{j=1}^{K} \left(\frac{1}{10}\right)^j = \frac{10 - (1/10)^{k+1}}{1 - 1/10} = \frac{10 - 1}{10^{k+1}}
\]

\[
< \frac{1}{9} \quad \forall \ k.
\]

\[
\therefore \ B = \left[2 + \frac{1}{9}, \infty\right) \quad & \quad C = (-\infty, -2 - 3 \cdot \frac{1}{9}]
\]

(c) The \(\inf A = \sup C = -2 - \frac{1}{3} \quad & \quad \sup A = \inf B = 2 + \frac{1}{9}\).

It is clear from how we got these values that both are indeed bounds, so to prove they are \(\inf(A) \ & \ \sup(A)\) we need only show that given any lower bound \(\gamma\) of \(A\)
\[
\gamma \leq -2 - \frac{1}{3} \quad & \quad \text{given} \ R \ \text{an upper bound of} \ A \quad R \geq 2 + \frac{1}{9},
\]
6c&d) cont.

First notice that given any \( n \in \mathbb{N}, \ 10^n \geq n. \)
This should be obvious. By theorem 2.14 (I think)
Given any \( x > 0 \) \( \exists \ n_0 \in \mathbb{N} \) st. \( \frac{1}{n_0} < x \), & since
\( n_0 \leq 10^{n_0} \) we find \( \frac{1}{10^{n_0}} < x \) also.

Now say \( r > -2 - \frac{1}{3} \) & \( r \) is a lower bound
for \( A \). Let \( x = r - (-2 - \frac{1}{3}) > 0 \) & let \( n_0 \in \mathbb{N} \)
be s.t. \( \frac{1}{10^{n_0}} < 3x \). By the above we have that

\[
\text{the \ (2n+1)^{\text{th}} \ element \ of \ A} = -2 - 3 \left( \frac{\sum_{j=1}^{n} \left( \frac{1}{10^j} \right)^j}{\sum_{j=1}^{n} \left( \frac{1}{10^j} \right)^j} \right) = -2 - 3 \left( \frac{\frac{1}{10} - \frac{1}{10^{n+1}}}{\frac{9}{10}} \right) = -2 - \frac{1}{3} \left( 1 - \frac{1}{10^{n_0}} \right) < -2 - \frac{1}{3} (1 - 3x) = -2 - \frac{1}{3} \frac{10}{3} + r - (-2 - \frac{10}{3}) = r.
\]

Thus \( r \) is not a lower bound for \( A \) & so \( -2 \frac{1}{3} = -2 - \frac{1}{3} = \inf(A) \).

Similarly, \( r \) is an upper bound for \( A \) with \( R < 2 + \frac{1}{9} = 2 \frac{1}{9} \)
Let \( x = 2 + \frac{1}{9} - R \) & let \( n_0 \in \mathbb{N} \) be s.t. \( \frac{1}{10^{n_0}} < 9x \).

We find that

\[
\text{the \ (2n+1)^{\text{th}} \ element \ of \ A} = 2 + \sum_{j=1}^{n} \left( \frac{1}{10^j} \right)^j = 2 + \frac{1}{9} \left( 1 - \frac{1}{10^{n_0+1}} \right) > 2 + \frac{10}{9} + x = 2 + \frac{1}{9} - (2 + \frac{1}{9} - R) = R.
\]

Thus \( R \) is not an upper bound for \( A \) & so \( 2 \frac{1}{9} = \sup A \).
(6e) We have \( BUC = (-\infty, -2 - \frac{1}{3}] \cup [2 + \frac{1}{9}, \infty). \)

Clearly \( 0 \notin BUC. \)

7. Given \( a > 0, \inf \frac{e^a}{n} \mid n \in \mathbb{N}^3 = 0. \)

**Proof:** \( \forall n \in \mathbb{N}, \frac{a}{n} > 0 \) so \( 0 \) is a lower bound for \( \frac{e^a}{n} \mid n \in \mathbb{N}^3. \) Suppose \( r > 0 \) is also a lower bound for \( \frac{e^a}{n} \mid n \in \mathbb{N}^3. \) Since \( a > 0, \frac{e^a}{a} > 0 \) & hence by theorem 2.4 (I'm sure that's it). \( \exists n \in \mathbb{N} \) s.t. \( \frac{1}{n_0} < \frac{e^a}{a} \). But then \( \frac{a}{n_0} \) is an element of \( \frac{e^a}{n} \mid n \in \mathbb{N}^3 \) & \( \frac{a}{n_0} > r, \) contradicting the assumption that \( r \) was a lower bound. Thus \( 0 \) is the greatest lower bound, i.e. \( 0 = \inf \frac{e^a}{n} \mid n \in \mathbb{N}^3. \)