Math 3283W: Week 1

- Syllabus and course page discussion

- Logic: when are complex statements/arguments true? Start with statements P, Q and build new statements.

\[ P \land Q \] (P and Q): true when both P and Q are true
\[ P \lor Q \] (P or Q): true when either P or Q is true (inclusive "or")
\[ \neg P \lor \neg Q \] (not P): true when P is false

These are only formally defined by truth tables. Two statements \( S, T \) are equivalent ("\( S \equiv T \)"") if and only if they have the same truth tables.

**Eq.** Truth table for \( \neg P \lor Q \):

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>( \neg P )</th>
<th>( \neg P \lor Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
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<td>T</td>
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</tbody>
</table>

(\( \neg P \lor Q \): False only if \( P \) is true and \( Q \) is false)

**Exercise.** Are \( \neg(P \land Q) \) and \( \neg P \lor \neg Q \) equivalent? Are \( \neg(P \lor Q) \) and \( \neg P \land \neg Q \) equivalent? (2nd one: left for the reader)

Rephrased first question: is \( \neg(P \land Q \text{ true}) \) the same as "either P false or Q false"?

**Truth table:**

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>( \neg P )</th>
<th>( \neg Q )</th>
<th>( P \land Q )</th>
<th>( \neg(P \land Q) )</th>
<th>( \neg P \lor \neg Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
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</tr>
</tbody>
</table>

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\( \text{same truth tables!} \) therefore, logically equivalent!
Exercise. Find an equivalent statement using "V" for \( \sim (P \land Q \land R) \). (Guess: \( \sim P \lor \sim Q \lor \sim R \); check it on your own.)

Exercise. Show that \( P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R) \) (distributive rule).

To define: \( P \Rightarrow Q \) ("P implies Q"). (Want: IF P true, then Q true. IF P false, no information for Q.)

We need to build \( P \Rightarrow Q \) from statements we already have. Define: \( P \Rightarrow Q \equiv \sim P \lor Q \) (refer to earlier truth table).

Equivalent forms:
1. \( P \Rightarrow Q \equiv \sim (P \land \sim Q) \)
2. \( P \Rightarrow Q \equiv \sim Q \Rightarrow \sim P \) (contrapositive form)

Proof of 1:

\[
\begin{align*}
P \Rightarrow Q & \iff \sim P \lor Q \\
& \iff \sim P \lor \sim (\sim Q) \\
& \iff \sim (P \land \sim Q)
\end{align*}
\]

(exercise from earlier)

Proof of 2:

\[
\begin{align*}
P \Rightarrow Q & \iff \sim P \lor Q \\
& \iff \sim (\sim Q) \lor \sim P \\
& \iff \sim Q \Rightarrow \sim P
\end{align*}
\]

Claim: \( P \Rightarrow Q \) is NOT equivalent to \( Q \Rightarrow P \).

Proof: Suppose \( P \) is false and \( Q \) is true. Then \( P \Rightarrow Q \) is true, but \( Q \Rightarrow P \) is false, so they cannot be equivalent.

Claim: \( P \equiv Q \) is the same as \( [P \Rightarrow Q] \land [Q \Rightarrow P] \) (sometimes written "\( P \Leftrightarrow Q \)"").

Proof: Suppose (1) \( P \Rightarrow Q \) and (2) \( Q \Rightarrow P \). Suppose \( P \) is true; then, by (1), \( Q \) is true; similarly, if \( Q \) is true, by (2) \( P \) is true. So \( P \) true if and only if \( Q \) true. Now suppose \( P \) is false. By the contrapositive form of (2), \( Q \) is false. Similarly, if \( Q \) is false, by the contrapositive form of (1), \( P \) is false. Thus \( P \) false if and only if \( Q \) false.

Intuition: IF \( P \Rightarrow Q \), \( P \) is a sufficient condition for \( Q \), and \( Q \) is a necessary condition for \( P \).

\[
(P \Rightarrow Q \land R) \equiv (P \Rightarrow Q) \land (P \Rightarrow R): \text{exercise}
\]

Logic and Proofs. Prove: If \( x \) is an integer and \( x^2 \) is odd, then \( x \) is odd. (Take the contrapositive!)

If \( x \) is even, then \( x^2 \) is even. But \( 2 \mid x \iff 2 \mid x^2 \).)

\[
\begin{align*}
\text{more detail: } x \text{ even } & \iff x = 2n \text{ for some } n \\
& \iff x^2 = 4n^2 \text{, and } 4n^2 \text{ is a multiple of } 2.
\end{align*}
\]

Exercise: If \( x \) is an integer, prove that \( x^2 \) is even if and only if \( x \) is \( \not\equiv \) even.

("Iff")
Quantifiers (universal: for all, \( \forall \); existential: there exists, \( \exists \)). These are used with variable statements:

Example: Let \( P(x) \) be "\( x > 7 \)" and \( Q(x) \) be "\( x^2 - 2x - 3 = 0 \)."

\((\forall x > 3)(P(x))\) in English reads "for all real \( x > 3 \), \( x > 7 \)" which is true.

\((\exists x < 0)(Q(x))\) "there exists \( x < 0 \) with \( x^2 - 2x - 3 = 0 \)" (true: try \( x = -1 \)).

However, \((\forall x > 2)(P(x))\) and \((\exists x > 4)(Q(x))\) are false.

De Morgan's laws: negating \( \land \) and \( \lor \):

\((1)\) \( \sim [\forall x \in A] \equiv \exists x \in A \sim P(x) \)

\((2)\) \( \sim [\exists x \in A] \equiv \forall x \sim P(x) \)

More statements:

\((3)\) \( \sim [\forall x \in A]P(x) \equiv \exists x (x \in A \land \sim P(x)) \)

\((4)\) \( \sim [\exists x \in A]P(x) \equiv (\forall x \in A)(\sim P(x)) \)

Proof of 3:

\( \sim [\forall x \in A]P(x) \equiv \exists x \sim P(x) \equiv \exists x \sim P(x) \equiv \exists x \sim P(x) \)

\( (4) \) is an exercise.

Example: Quantify:

(1) Every positive number less than 1 is less than its square root.

(2) Every positive number is greater than the square of some number.

Also (3): if \( P(x) \) is "\( y^2 < x \)" is \( \sim [\forall x > 0 \exists x > 0 P(x)] \) true?

(1): \( \forall x > 0 (x < 1 \implies x^2 < x) \)

(2): \( \forall x > 0 \exists y > 0 y^2 < x \) (true: just take \( 0 < y < \sqrt{x} \))

(3): this is the negation of what we just proved in (2), so false.

Exercise. Every positive number is greater than the cube of some number. (Quantify and prove)

Let \( \mathbb{Z}^+ = \{0, 1, 2, \ldots\} \) = non-negative integers. Are the following statements true?

(1) \( \forall n \in \mathbb{Z}^+, n^2 + n + 37 \) is prime \( \text{false} \) \( n = 1 \implies n^2 + n + 37 = 39 = 3 \cdot 13 \) (true for \( n = 0) \)

(2) \( \forall n \in \mathbb{Z}^+, n^2 + n + 31 \) is prime \( \text{false} \) \( n = 1 \implies n^2 + n + 31 = 33 = 11 \cdot 3 \) (true for \( n = 0) \)

(3) \( \forall n \in \mathbb{Z}^+, n^2 + n + 41 \) is prime \( \text{false} \) \( n = 0 \implies 41 \), \( n = 2 \implies 47 \) \( n = 3 \implies 53 \) \( n = 4 \) all primes.

We need something better. Try \( n = 41 \): \( 41^2 + 41 + 41 = 41 \cdot 43 \) is not prime, so (1) is false.
Exercise. Let $p$ be a prime number. Then it's not true that $n^2 + n + p$ is prime for all $n \in \mathbb{Z}^+$. If $m$ is any integer, it's not true that $n^2 + mn + p$ is prime for all $n \in \mathbb{Z}^+$. (stated ambiguously in lecture)

Functions and quantifiers: Let $f: \mathbb{R} \to \mathbb{R}$ be a function. Let $P(x)$ be "$f$ is continuous at $x$". Quantify the following:

1. $f$ has a maximum value
2. $f$ does not have a minimum value
3. $f$ is continuous on $[a,b]$, then $f$ has both a maximum and minimum value on $[a,b]$.

Answers:
1. $\exists y \in \mathbb{R} \forall x \in \mathbb{R} (f(y) > f(x))$
2. $\forall y \in \mathbb{R} \exists x \in \mathbb{R} (f(y) > f(x))$
3. To say "$f$ is continuous on $[a,b]$ is to say $\forall x \in [a,b], f(x)$ is continuous at $x$".

To say "$f$ has both a maximum and minimum value on $[a,b]$" is to say $\exists y, w \in [a,b]$ such that $\forall x \in [a,b], f(w) \leq f(x) \leq f(y)$.

Putting them together, $\forall x \in [a,b], f(x)$ is continuous at $x$ and $\exists y, w \in [a,b] \forall x \in [a,b], f(w) \leq f(x) \leq f(y)$.

Is this the same as \textit{factoring out $w$}?  No...