4/12/2010
Power series (cont.)

Examples:

(1) \( f(x) = (1 + x)^{\frac{1}{2}} \)

\[
\begin{align*}
    f'(x) &= \frac{1}{2}(1 + x)^{-\frac{1}{2}} \\
    f''(x) &= \frac{1}{2} \cdot (-\frac{1}{2})(1 + x)^{-\frac{3}{2}} \\
    f'''(x) &= \frac{1}{2} \cdot (-\frac{1}{2}) \cdot (-\frac{3}{2})(1 + x)^{-\frac{5}{2}} \\
    f^{(4)}(x) &= -\frac{1 \cdot 3 \cdot 5}{2^4}(1 + x)^{-\frac{7}{2}} \\
    &\vdots \\
    f^{(n)}(x) &= (-1)^{n+1}\frac{1 \cdot 3 \cdot 5 \cdots (2n - 3)}{2^n}(1 + x)^{-\frac{2n+1}{2}} \\
\end{align*}
\]

So \( f^{(n)}(0) = (-1)^{n+1}\frac{1 \cdot 3 \cdot 5 \cdots (2n - 3)}{2^n} \) and

\[
TS = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}1 \cdot 3 \cdot 5 \cdots (2n - 3)}{2^n} x^n
\]

(2) \( f(x) = \sin x \) at \( x = \frac{\pi}{2} \).

\[
\begin{align*}
    f^{(n)}(x) : \quad &\sin x \quad \cos x \quad -\sin x \quad -\cos x \quad \sin x \quad \cdots \\
    f^{(n)}(\frac{\pi}{2}) : \quad &1 \quad 0 \quad -1 \quad 0 \quad 1 \quad \cdots \\
\end{align*}
\]

So

\[
T(\frac{\pi}{2}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x - \frac{\pi}{2})^{2n}
\]

We have shown that the Taylor series of \( \sin x \) at \( x = 0 \) is

\[
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!}
\]
From this example, we can see that the Taylor series of a function at different points can be very different. Moreover,

\[ T_3(0) = x - \frac{x^3}{3!}, \quad T_3\left(\frac{\pi}{2}\right) = 1 - \frac{(x - \frac{\pi}{2})^2}{2!} \]

These two polynomials even have different degrees!

(3) Let \( f(x) = a_0 + a_1x + \cdots + a_nx^n \). \( f(x) \) is a polynomial and

\[ f(0) = a_0, \quad f'(0) = a_1, \quad f''(0) = 2a_2 = 2!a_2, \cdots \]

Note that \( f^{(m)}(x) = 0 \) if \( m > n \). So

\[ T(0) = a_0 + \cdots + a_nx^n \]

It means that any polynomial of \( x \) is its Taylor series.

(4) Consider \( f(x) = e^{x^2} \). Since \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) holds for all real \( x \), replace \( x \) with \( x^2 \) can get

\[ e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \]

If we directly compute \( T_n(0) \), we have

\[ f(0) = 1, \quad f'(0) = 2xe^{x^2}|_{0} = 0, \quad f''(0) = 2e^{x^2} + 4xe^{x^2}|_{0} = 2, \cdots \]

and hence \( T_0(0) = 1, T_1(0) = 1, T_2(0) = 1 + x^2, \cdots \), i.e. the above series is the Taylor series for \( e^{x^2} \).

(5) Consider \( P(x) = 2 + x - x^3 + x^5 \). It is clear that \( T_3(0) = 2 + x - x^3 \), we want to find \( T_3(2) \).

We know that

\[ T_3(2) = P(2) + P'(2)(x - 2) + \frac{P''(2)}{2!}(x - 2)^2 + \frac{P'''(2)}{3!}(x - 2)^3 \]

\[ P(2) = 2 + 2 - 8 + 32 = 28 \]
\[ P'(2) = 1 - 3 \cdot 4 + 5 \cdot 16 = 69 \]
\[ P''(2) = -3 \cdot 2 \cdot 2 + 5 \cdot 4 \cdot 8 = 148 \]
\[ P'''(2) = -6 + 60 \cdot 4 = 234 \]

So

\[ T_3(2) = 28 + 69(x - 2) + 74(x - 2)^2 + 39(x - 2)^3 \]
Differentiating and Integrating Power Series

We now show how to differentiate and integrate power series by doing the obvious way: computing term-by-term. This will allow us to find the power series for many more functions. A key result is

**Theorem 0.1.** If \( \sum_{k=0}^{\infty} a_k x^k \) converges on \((-c, c)\), then the power series formula by term-by-term differentiation

\[
\sum_{k=0}^{\infty} \frac{d}{dx} (a_k x^k) = \sum_{k=1}^{\infty} k a_k x^{k-1} \quad \text{(Remark: k starts from 1)}
\]

also converges on \((-c, c)\).

**Proof.** Pick \( t \in (-c, c) \) and \( \varepsilon > 0 \) s.t. \(|t| < |t| + \varepsilon < c\). Since \(|t| + \varepsilon < R\) (R: the radius of convergence), \( \sum |a_k (|t| + \varepsilon)^k| \) converges. Now

\[
(k|t|^{k-1})^{\frac{1}{k}} = k^{\frac{1}{k}} |t|^{1-\frac{1}{k}} \xrightarrow{k \to +\infty} 1 \cdot |t| = |t|
\]

So \( \exists k_0, k \geq k_0 \Rightarrow (k|t|^{k-1})^{\frac{1}{k}} < |t| + \varepsilon \) i.e. \( k|t|^{k-1} < (|t| + \varepsilon)^k \)

Hence, by CT, \( \sum_{k=k_0}^{\infty} |a_k k| |t|^{k-1} = \sum_{k=k_0}^{\infty} |ka_k k^{k-1}| \) converges and so does \( \sum_{k=1}^{\infty} |ka_k k^{k-1}|. \)

**Corollary 0.2.**

1. \( \sum a_k x^k \) and \( \sum ka_k x^{k-1} \) have the same radius of convergence \( R \).

2. If \( f(x) = \sum a_k x^k \), then \( f'(x) = \sum ka_k x^{k-1} \).

**Proof.**

1. From theorem 0.1, we know that \( \sum ka_k x^{k-1} \) converges on \((-R, R)\). Now suppose \( R_1 \) is the radius of converges for \( \sum ka_k x^{k-1} \) and \( |x_1| < R_1 \). Then \( \sum |ka_k x_1^{k-1}| \) converges. Since \( |a_k x_1^{k-1}| \leq |ka_k x_1^{k-1}| \), we know that \( \sum |a_k x_1^{k-1}| \) converges, as well as \( |x_1| \sum |ka_k x_1^{k-1}| = \sum |a_k x_1^k| \). So \( R_1 \leq R \), which means that \( R_1 = R \).

2. Skip proof.

We know can compute all derivatives of \( f(x) \) in the same way.

\[
f''(x) = \sum_{k=0}^{\infty} \frac{d^2}{dx^2} (a_k x^k) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}
\]

\[
f'''(x) = \sum_{k=3}^{\infty} k(k-1)(k-2) a_k x^{k-3} \text{ etc}
\]

4/14/2010
Enrichment: partial sum of divergent positive series.

Just as rearrangements of conditional convergent series giving any real numbers, divergent series have similar properties for summing rearranged terms. Note that \( \sum_{k=1}^{\infty} \frac{1}{n} \) diverges. We have:

Any positive rational number can be written as a finite sum of terms from \( \sum_{k=1}^{\infty} \frac{1}{n} \).

Examples:

(1) \( \frac{27}{37} \).

Find smallest \( n_1 \) s.t. \( \frac{1}{n_1} < \frac{27}{31} \). \( n_1 = 2 \) because \( \frac{1}{2} < \frac{27}{31} \) and \( \frac{1}{1} > \frac{27}{31} \).

If we look at \( \frac{27}{31} - \frac{1}{2} = \frac{54-31}{62} = \frac{23}{62} \), then \( n_2 = 3 \) since \( \frac{1}{3} \leq \frac{23}{62} \).

\[ \frac{23}{62} - \frac{1}{3} = \frac{69-62}{186} = \frac{7}{186} \]. Since \( \frac{186}{7} = 26.57 \), we know that \( n_3 = 27 \) and \( \frac{1}{186} = \frac{27}{2322} = \frac{1}{332} \).

So \( n_4 = 774 \) and we are done:

\[
\frac{27}{31} = \frac{1}{2} + \frac{1}{3} + \frac{1}{27} + \frac{1}{774}
\]

(2) \( \frac{3}{7} \).

\( n_1 = 3 \) since \( \frac{1}{3} < \frac{3}{7} \) and \( \frac{3}{7} - \frac{1}{3} = \frac{9-7}{21} = \frac{2}{21} \).

\( n_2 = 311 \) since \( \frac{1}{11} < \frac{2}{21} \) and \( \frac{2}{21} - \frac{1}{11} = \frac{22-21}{231} = \frac{1}{231} \).

\( n_3 = 231 \) and \( \frac{3}{7} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231} \).

Check:

\[
\frac{1}{3} + \frac{1}{11} + \frac{1}{231} = \frac{77 + 21 + 1}{231} = \frac{99}{231} = \frac{9}{21} = \frac{3}{7}
\]

Note: the sum is unique by fact in number theory.

Exercises: Do the above for \( x = \frac{5}{3} \) and \( x = \frac{28}{41} \).

(3) \( \frac{10}{7} = 1 + \frac{3}{7} \).

We know that \( \frac{3}{7} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231} \), we can’t use 3, 11 and 231 to compute 1. We start with \( \frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{19}{20} < 1 \) which is the largest possible partial sum of the series \( \sum_{k=1}^{\infty} \frac{1}{n} \) removing \( \frac{1}{3}, \frac{1}{11}, \frac{1}{231} \) and less than 1. Then \( 1 - \frac{19}{20} = \frac{1}{20} \) and we have \( \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{20} \).

Thus

\[
\frac{10}{7} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{11} + \frac{1}{20} + \frac{1}{231}
\]

Exercises: Do the above for \( x = \frac{5}{3} \) and \( x = \frac{28}{9} \).
(4) The above works for the odd harmonic series \(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\)
and rational number \(\frac{p}{q}\) with \(q\) odd. For example, if \(x = \frac{7}{13}\),
the largest \(\frac{1}{n}\) with \(n\) odd and less than \(\frac{7}{13}\) is \(\frac{1}{3}\), so \(n_1 = 3\) and
\[
\frac{7}{13} - \frac{1}{3} = \frac{21-13}{39} = \frac{8}{39},
\]
\(\frac{1}{5} < \frac{8}{39}\) and \(n_2 = 5\), \(\frac{8}{39} - \frac{1}{5} = \frac{40-39}{195} = \frac{1}{195}\).
So \(n_3 = 195\) and we have \(\frac{7}{13} = \frac{1}{3} + \frac{1}{5} + \frac{1}{195}\).
For full harmonic series, we have \(\frac{7}{13} = \frac{1}{2} + \frac{1}{26}\).

Integration of power series

Consider a PS \(\sum_{k=0}^{\infty} a_k x^k\) and the PS formed by term-by-term integration:
\[
\sum_{k=0}^{\infty} \int a_k x^k \, dx = \sum_{k=0}^{\infty} \frac{a_k x^{k+1}}{k+1}
\]
If \(R\) is the radius of converges for this new series, then since \(\frac{d}{dx}(\frac{a_k x^{k+1}}{k+1}) = a_k x^k\), the original series also has radius of convergence \(R\) as before. If
\(f(x) = \sum_{k=0}^{\infty} a_k x^k\), then
\[
\int f(x) \, dx = \sum_{k=0}^{\infty} \int a_k x^k \, dx = \sum_{k=0}^{\infty} \frac{a_k x^{k+1}}{k+1}
\]
\(= C + a_0 x + a_1 x^2 + a_2 \frac{x^3}{3} + \cdots\), \(C\): constant of integration

Examples:

(1) We have
\[
\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3} + \frac{x^5}{5!} - \cdots
\]

Then
\[
\cos x = \frac{d}{dx}(\sin x) = \sum_{n=0}^{\infty} \frac{d}{dx}((-1)^n \frac{x^{2n+1}}{(2n+1)!}) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots
\]

(2) Consider
\[
\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x - 1)^n, 0 < x \leq 2
\]
Replace \(x - 1\) by \(w\). Since \(0 < x \leq 2, -1 < x - 1 \leq 1\). So \(-1 < w \leq 1\) and we get \(\ln(1 + w) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} w^n\) with
\( R = 1 \) as before, but now it is a PS with \( w \) at \( w = 0 \). We can form a PS for
\[
\ln\left(\frac{1+x}{1-x}\right) = \ln(1 + x) - \ln(1 - x) = \sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{n} - a_n\right)x^n
\]
where \( \sum_{n=1}^{\infty} a_n x^n = \ln(1 - x) \). But
\[
\ln(1 - x) = \int \frac{-1}{1-x} \, dx = (-1) \sum_{n=0}^{\infty} \int x^n \, dx = (-1) \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} -\frac{x^n}{n} \quad (C = 0)
\]
So \( \frac{(-1)^{n+1}}{n} - \frac{1}{n} = \begin{cases} \frac{2}{n}, & n = 2k + 1 : odd \\ 0, & n = 2k : even \end{cases} \) and \( R = 1 \).
\[
\ln\left(\frac{1+x}{1-x}\right) = \sum_{k=0}^{\infty} \frac{2}{2k+1} x^{2k+1} = 2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}
\]
This allows us to estimate \( \ln w \) for any positive \( w \). If we solve \( \frac{1+x}{1-x} = w \) for \( 0 < x < 1 \),
\[
1 + x = w(1 - x) = w - wx, \quad (1 + w)x = w - 1, \quad x = \frac{w - 1}{w + 1}
\]
\( w = 9, x = .8 \),
\[
\ln 9 \approx 2\left[\frac{8}{1} + \frac{8^3}{3} + \frac{8^5}{5} \right] \approx 2 \cdot 1.0362 = 2.0724
\]
(A modest estimate is 2.1972. Adding \( \frac{8^7}{7} \) gives 2.1023)
\( w = 2, x\frac{1}{3} \),
\[
\ln 2 \approx 2 \times \left[\frac{1}{3} + \frac{1^3}{3} + \frac{1^5}{5} \right] = 2 \times .3465 = .6930
\]
an excellent estimate \((\ln 2 \approx .69315)\).
(3) We have \( e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \) (start \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \), and replace \( x \) by \( x^2 \)). Suppose we want to estimate \( \int_0^1 e^{x^2} \, dx \). Now
\[
\int e^{x^2} \, dx = \sum_{n=0}^{\infty} \int \frac{x^{2n}}{n!} \, dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1) \cdot n!} = x + \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!} + \cdots
\]
So \( \int_0^1 e^{x^2} \, dx \approx x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} |_{x=0} = 1 + \frac{1}{3} + \frac{1}{10} + \frac{1}{42} = 1.4571 \)

4/16/2010
The geometric series together with differentiation/integration of power series gives us lots of new PS. Since we know

$$\frac{1}{1 + x^2} = \frac{1}{1 - (-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

We can get

$$\arctan x = \int \frac{1}{1 + x^2} dx = \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + C$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad (C = \arctan 0 = 0)$$

This PS converges for $R = 1$.

At $x = -1$, we get $\sum_{n=0}^{\infty} (-1)^n (-1)^{2n+1} = \sum_{n=0}^{\infty} (-1)^{n+1}$ which converges by alternating series test.

At $x = 1$, we get $\sum_{n=0}^{\infty} (-1)^n$ which also converges .

**Theorem 0.3** (Abel Theorem). *(see textbook)*

If $f(x) = \sum_{n=0}^{\infty} a_k x^k$ converges for $|x| < 1$ and $\sum_{n=0}^{\infty} a_k$ converges, then $f(1) = \sum_{n=0}^{\infty} a_k$ when $f(1)$ is continuous at $x = 1$.

Thus $\pi/4 = \arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

(Note: this is amazing because an irrational number $\pi/4$ can be expressed as sum of rational numbers. But this PS is not useful because it converges too slowly.)

**Example:** Find power series for $f(x) = \frac{1}{1+x^2}$.

We can multiply PS of $\frac{1}{1+x^2}$ to get it. An alternate way is to use the following formula:

$$\frac{d}{dx} (\frac{1}{1 + x^2}) = (-1) \frac{1}{(1 + x^2)^2} \cdot 2x = \frac{-2x}{(1 + x^2)^2}$$

Since $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$,

$$\frac{d}{dx} (\frac{1}{1 + x^2}) = \sum_{n=1}^{\infty} (-1)^n 2n \cdot x^{2n-1} = \sum_{m=0}^{\infty} (-1)^{m+1} (m+1) \cdot x^{2m+1} \quad (m = n-1)$$
So \[ \frac{-2x}{(1 + x^2)^2} = 2 \sum_{n=0}^{\infty} (-1)^{n+1}(n + 1) \cdot x^{2n+1} \]
\[ \frac{x}{(1 + x^2)^2} = \sum_{n=0}^{\infty} (-1)^n(n + 1) \cdot x^{2n+1} \]
\[ \frac{1}{(1 + x^2)^2} = \sum_{n=0}^{\infty} (-1)^n(n + 1) \cdot x^{2n} = 1 - 2x^2 + 3x^4 - 4x^6 + \cdots \]

Test: when \( x = .17, \frac{1}{(1+x^2)^2} \approx .9446 \) and the first 4 terms \( \approx .9536 \). So the error \( \approx .009 \).

**Counting sets: levels of infinity**

Recall: \( A \neq \emptyset \) is finite if \( \exists n \in \mathbb{N} \) such that \( A \) has precisely \( n \) elements:

\[ A = \{a_i | 1 \leq i \leq n\} \quad & \quad \forall i \neq j \Rightarrow a_i \neq a_j \]

Another way: if \( \mathbb{N}_n = \{1, 2, \cdots, n\} \), then there is an 1-to-1 and onto function \( f : \mathbb{N}_n \rightarrow A, f(i) = a_i \). (Onto means that the set \( \text{Range} f = \{y | y = f(x) \text{ for some } x\} \) is equal to \( A \).) 1-to-1 and onto functions are also called bijections.

**Definition 0.4.** \( f : C \rightarrow D \) is 1-to-1 if

\[ \forall c_1, c_2 \in C, c_1 \neq c_2 \Rightarrow f(c_1) \neq f(c_2). \]

\( f : C \rightarrow D \) is onto if

\[ \forall d \in D, \exists c \in C \Rightarrow f(c) = d \]

This is the corresponding definition of finite:

\( A \neq \emptyset \) is finite if \( \exists n \in \mathbb{N} \) and a bijection \( f : \mathbb{N}_n \rightarrow A \).

**Examples:**

1. \( f_j, g_j : \mathbb{N} \rightarrow \mathbb{N}, f_j(n) = n + j, g_j(n) = jn \) where \( j \in \mathbb{N} \) is fixed. (For instance, \( f_2(n) = n + 2, g_2(n) = 2n \).) Both are 1-to-1, but not onto:
   - If \( j \neq 1 \), suppose \( f_j(n_1) = f_j(n_2), \) so \( n_1 + j = n_2 + j \) and \( n_1 = n_2 \).
   - This is the contrapositive of \( n_1 \neq n_2 \Rightarrow f_j(n_1) \neq f_j(n_2) \). So \( f_j \) is 1-to-1.
   - Since \( jn_1 = jn_2 \rightarrow n_1 = n_2, \) \( g_j \) is also 1-1.
   - If \( j \neq 1 \), \( f_j(\mathbb{N}) = \{n > j | n \in \mathbb{N}\} = \{1 + j, 2 + j, 3 + j, \cdots\} \) and \( 1, 2, \cdots, j \notin f_j(\mathbb{N}) \). So \( f_j \) is not onto. (Note: in this case, it is true even if \( j = 1 \).)
   - \( g_j(\mathbb{N}) = \{jn | n \in \mathbb{N}\} = \{j, 2j, 3j, \cdots\} \) and \( 1, 2, \cdots, j - 1 \notin \)
$g_j(N)$. So $g_j$ is not onto.

If $j = 1$, $g_j(n) = 1 \cdot n = n$ and $g_1$ is onto (so a bijection).

(2) Suppose $A$ is a finite set and $f : \mathbb{N}_n \rightarrow A, f(j) = a_j$ a bijection.
Then $g : \mathbb{N}_n \rightarrow A, g(j) = a_{n+1-j}$ is also a bijection.

We can use these ideas to refine our notion of infinitity (previous definition: not finite.)

**Definition 0.5.** $A$ is countably infinite if $\exists f : \mathbb{N} \rightarrow A$ such that $f$ is a bijection. Setting $f(i) = a_i$, we can express $A = \{a_i| i \in \mathbb{N}\}$ and $i \neq j \Rightarrow a_i \neq a_j$.

**Example:** The set $O$ of odd positive integers and set $E$ of even positive integers are countably infinite. The functions $f : \mathbb{N} \rightarrow E, f(n) = 2n$ and $g : \mathbb{N} \rightarrow O, g(n) = 2n - 1$ are bijections (Exercise).