Prop: Let $A \& B$ be finite sets. Then $A \times B$ is finite and $\#(A \times B) = \#(A) \cdot \#(B)$

Proof: Let $A = a_1, \ldots, a_m$ and $B = b_1, \ldots, b_n$, list and count the elements of $A \times B$.

Theorem: Let $A \& B$ be arbitrary sets, then

At least one of the sets $A$ or $B$ infinite $\implies$ $A \cup B$ is infinite.

Proof: (Contradiction)

Examples

1. $A = \{1, 2, 3\}, B = \mathbb{N}$, $A \cup B = \mathbb{Z}_{\geq 0}$

2. Suppose $A \cup B = \mathbb{Z}$ & $A$ is a finite subset of $\mathbb{Z}$, then $B$ MUST be infinite.

Set identities (prove these!)

Let $A, B \& C$ be arbitrary sets

1. $A = (A \setminus B) \cup (A \cap B)$

2. $A \cup B \cup C = (A \cup B) \cup (B \cap C) \cup (C \setminus A) \cup (A \cup B) \cup (B \cap C) \cup (C \setminus A)$

3. Suppose $A, B, C$ are pairwise disjoint (i.e., $A \cap B = B \cap C = C \setminus A = \emptyset$) then $A \cup B \cup C = \emptyset$. The converse, however, is FALSE.

4. If $A, B, C$ are pairwise disjoint, then $\#(A \cup B \cup C) = \#(A) + \#(B) + \#(C)$

Induction

Lemma: Suppose that $A \subseteq \mathbb{Z}_{\geq 0}$ s.t. (1) $0 \in A$ &

(2) $\forall n \in \mathbb{Z}_{\geq 0}$ if $n \in A$ then $n+1 \in A$.

Then $A = \mathbb{Z}_{\geq 0}$. Proof by contradiction (hint let $p$ be the smallest element of $\mathbb{Z}_{\geq 0} \setminus A$).
Principal of Mathematical induction

Let \( P(n) \) be a statement which depends on \( n \in \mathbb{Z}^+ \). Suppose that

1. \( P(1) \) is true
2. \( \forall n \in \mathbb{Z}^+ (P(n) \implies P(n+1)) \)

Then \( P(n) \) is true \( \forall n \in \mathbb{Z}^+ \).

Proof hint. Consider the set \( A_p := \{ n \in \mathbb{Z}^+: P(n) \text{ is true} \} \). Can we apply lemmas above?

Corollary: Let \( Q(n) \) depend on \( n \in \mathbb{Z}^+ \). Let \( m \in \mathbb{Z}^+ \) be fixed.

Suppose that

1. \( Q(m) \) is true
2. \( \forall n \geq m (Q(n) \implies Q(n+1)) \)

Then \( Q(n) \) is true \( \forall n \in \mathbb{Z}^+ \) s.t. \( n \geq m \).

Example 65

1. \( 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \)
2. \( 1^3 + 2^3 + \ldots + n^3 = \left( \frac{n(n+1)}{2} \right)^2 \)
3. \( \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n(n+1)}{2n^2} \)
4. \( P(n) := "The \text{ sum of the first } n \text{ even numbers is } n(n+1)" \)
5. \( 1 + 2 + \ldots + n = \frac{n(n+1)}{2} \) (do this w/ induction using (4))
Jan 25

\[ P(x) := \text{"f(x) is continuous at } x \text{"} \]

Consider \( Q := \text{"f has a max & min on } [a,b] \text{"} \)

(What is this in terms of quantifiers? Eq \( \text{"f has a max on } [a,b] \text{"} \) becomes
\[ \exists x \in [a,b] \text{ s.t. } \forall y \in [a,b] f(x) \leq f(y) \]

The sentence \( \text{"f(x) is continuous on } [a,b] \text{"} \) is given in terms of quantifiers by \( \forall x \in [a,b], P(x) \), so what about the following theorem from calculus?

Thm: If \( f(x) \) is a continuous function on \( [a,b] \), then \( f \) has a maximum & a minimum on \( [a,b] \).

Does this translate to
\[ \forall x \in [a,b], (P(x) \Rightarrow \exists y \in [a,b], \exists z \in [a,b], f(x) \leq f(y) \leq f(z)) \]?

If not, how are they different and why?

Ex: Let \( P(x) := \text{"x+4 \geq 0\"} \). The statements
(i) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t. } P(xy) \)
(ii) \( \forall y \in \mathbb{R}, \exists x \in \mathbb{R} \text{ s.t. } P(xy) \)

These statements are NOT equivalent (Proof?)

Finite/Infinite sets

Def: A set \( S \) is finite if \( S = \emptyset \) or if \( \exists n \in \mathbb{N} \text{ s.t. } S \) has exactly \( n \) elements; a set is called infinite if it is not finite. For a finite set \( S \) define the cardinality of \( S \) (denoted \( \#(S) \)) to be 0 if \( S = \emptyset \) or else \( \#(S) = n \) where \( n \in \mathbb{N} \) is the number of elements in \( S \).

Examples: The following are infinite sets
(i) \( \exists \text{prime numbers, even numbers, prime numbers, integers which are multiples of } \sqrt{3} \)
(Where \( n \) is some non-zero integer), \( \mathbb{Z}, \mathbb{Q}, \text{ etc.} \)
Notice that \( \{ \text{odd} \} \cap \{ \text{primes} \} \) is infinite, but \( \{ \text{odd} \} \cap \{ \text{evens} \} = \emptyset \) is finite. In particular, the intersection of two infinite sets may or may not be infinite.

Claim: The union of two infinite sets is infinite. (Proof by contradiction)

Prop: If \( A \) & \( B \) are finite sets, then

1. \( A \cup B \) & \( A \cap B \) are finite.
2. \( \#(A \cup B) \leq \#(A) + \#(B) \)
3. \( \#(A \cup B) = \#(A) + \#(B) \iff A \cap B = \emptyset \)

Exercise: If \( A, B, C \) are finite sets, then \( A \cup B \cup C \) is finite & \( \#(A \cup B \cup C) \leq \#(A) + \#(B) + \#(C) \).

Note: \( A \cap B \cap C = \emptyset \) is NOT enough to deduce that \( \#(A \cup B \cup C) = \#(A) + \#(B) + \#(C) \). What is the necessary condition?

Theorem: For any pair of finite sets \( A \) & \( B \),

\[ \#(A \cup B) + \#(A \cap B) = \#(A) + \#(B) \]

Proof Hint: Note \( A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B) \) & \( (A \setminus B) \cap (B \setminus A) = \emptyset \)
Jan 29

What's wrong with the following argument?

Let \( P(n) = \text{"The total number of legs in a group of n horses is always odd"} \)

Suppose \( P(n) \) is true for some \( n \), then

\[
\#(\text{legs in a group of } n \text{ horses}) = \#(\text{legs on a horse}) + \#(\text{legs in a group of } n \text{ horses}) = 4 + \text{odd number} = \text{odd number}
\]

hence \( P(m) \) is true. Thus \( (P(n) \Rightarrow P(n+1)) \) for any \( n \in \mathbb{Z}^{+} \).

So \( P(n) \) is true. Hence \( \mathbb{Z}^{+} \)  

Prove: "For any convex \( n \)-gon \( A \), the sum of the interior angles of \( A \) must be \( \pi(n-2) \)" is true \( \forall n \geq 3 \).

\( P(3) \): sum of interior angles in a triangle is \( \pi \). Suppose \( P(n) \) is true. Let \( A \) be the \( n \)-gon drawn below.

Draw a chord from \( n \) to \( 1 \), then we get \( A \) decomposed as a triangle \( T \) and a \( (n-1) \)-gon \( A' \).

Note: Sum of angles of \( A = (\text{sum of angles of } T) + (\text{sum of angles of } A') = \pi + \pi(n-2) = \pi(n+1) - \pi \).

Theorem: \( \mathbb{Z}^{+} \) is well-ordered. That is to say, every non-empty subset \( S \subseteq \mathbb{Z}^{+} \) has a minimal element.

Proof by contradiction: noting for any \( b \in S \) the set

\[
S \times \mathbb{Z}^{+} \text{ is finite.}
\]

Triangular Numbers

\[
T(n) = \frac{n(n+1)}{2}
\]

\[
T(n) = 1 + \ldots + n = \frac{n(n+1)}{2}
\]

Dots in a dotted triangle with \( n \)-rows

\[
\begin{align*}
\text{for } n = 1 & : 1 \\
\text{for } n = 2 & : 4 \\
\text{for } n = 3 & : 9 \\
\text{for } n = 4 & : 16 \\
\text{for } n = 5 & : 25 \\
\text{for } n = 6 & : 36
\end{align*}
\]
Pentagonal numbers

\[ P(n) = \# \text{ of dots in a dotted pentagon with } n \text{-layers} \]

Claim \( P(n+1) = 4 + P(n) + 3(n-1) \quad \forall n \geq 1 \)

Proof

\[
P(n) = P(n) - P(n-1) + P(n-1) - P(n-2) + P(n-2) - \ldots - P(1) + P(0) \\
= 4 + 3(n-2) + 4 + 3(n-3) + \ldots + 4 + 1 \\
= 4n + 3 \left( \frac{n(n-1)}{2} \right) 
\]