2/1/2010

Proof without words: picture depicts
What is being proved from Fig.3.1?

1 Adding more and more dots gives bigger and bigger squares.
   → It is too vague and it is not actually a math statement.
2 Each consecutive line has two more dots than the previous line.
   → Nothing to prove.
3 The sum of consecutive odd numbers gives a square number .
   It can be proved by induction:

Observation: \( P(n) : 1 + 3 + \cdots + (2n + 1) = ? \)

\[
\begin{align*}
n = 1, & \quad 1 + (1 + 2) = 1 + (1 + 2 \cdot 1) = 4 = 2^2 \\
n = 2, & \quad 1 + (1 + 2) + (1 + 4) = 1 + (1 + 2 \cdot 1) + (1 + 2 \cdot 2) = 9 = 3^2
\end{align*}
\]

So we guess that \( P(n) \) is \( 1 + 3 + \cdots + (2n + 1) = (n + 1)^2 \) and prove it by induction.

\( n = 0 \), OK.

Assume \( P(n) \) is true, we want to show that \( P(n + 1) \) is true.

\[
\begin{align*}
P(n) &= (n + 1)^2 + 2n + 3 \\
&= n^2 + 2n + 1 + 2n + 3 \\
&= n^2 + 4n + 4 \\
&= (n + 2)^2 \\
&= ((n + 1) + 1)^2
\end{align*}
\]

So \( P(n + 1) \) is true.

**Upper and lower bounds**

Suppose \( A(\neq \emptyset) \subset \mathbb{R} \) has an upper bound (bounded above). Let

\[
B = \{ r \in \mathbb{R} | r : \text{upper bound for} \ A \} \neq \emptyset
\]
Suppose $B$ has a smallest element $w$. Then $w$ is called the least upper bound of $A$, or the supremum of $A$, write $w = \text{lub}A$ or $w = \text{sup}A$. Thus $w = \text{sup}A$ if

1. $w$ is an upper bound for $A$ and
2. if $r$ is an upper bound for $A$, then $r \geq w$.

Another form of (2), using contrapositive:

$$\forall r \in \mathbb{R} (r < w \Rightarrow \exists a \in A, r < a)$$

Note that any $r < w$ is not an upper bound.

Facts

- If $r > w = \text{sup}A$, then $r \in B$.
  Since $w$ is also an upper bound, $B$ is a ray $[w, \infty)$.
- Let $\varepsilon > 0$, then $w - \varepsilon < w$ and by $(2')$, $\exists a \in A, w - \varepsilon < a$, so we also have an equivalent condition:
  $$\forall \varepsilon > 0, \exists a \in A, w - \varepsilon < a.$$
(1) $A = (0, 1)$. Then $\sup(0, 1) = 1$ and $\inf(0, 1) = 0$.  
(Observe: 1 is an upper bound. If $r < 1$, we have to show that \( r \) is not an upper bound. Or we want to find \( s \in (0, 1) \) such that \( r < s < 1 \). Choose average \( \frac{1+r}{2} \), then \( r < \frac{1+r}{2} = s < 1 \).) 
\( A' = [0, 1] \) also has $\sup[0, 1] = 1$ and $\inf[0, 1] = 0$. So $\sup A$ and $\inf A$ may or may not be an element of $A$. 
\([-\infty, 0] \quad (0, 1) \quad [1, \infty) \)
\[ \uparrow \quad \uparrow \quad \uparrow \]
set of lower bounds \( A \) set of upper bounds

(2) 
\[ A = \{ \frac{1}{2^n} : n \in \mathbb{N} \} = \{ \frac{1}{n} | n \in \mathbb{N} \} \]
Since $\frac{1}{n}$ is decreasing, 1 is the largest element of $A$ and $\sup(A) = 1$.  

(3) 
\[ A = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots \right\} \]
A is bounded above by 1. What is $\sup(A)$? Note that 
\[ \frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1} \]
Let $\varepsilon > 0$, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n+1} < \varepsilon$. Then 
\[ 1 - \varepsilon < 1 - \frac{1}{n+1} = \frac{n}{n+1} \]
By (2'), $1 = \sup A$.

(4) Let 
\[ A = \{ x | x^3 < 4 \} \]
Now $[x > 0 \text{ and } x^3 < 4]$ iff $0 < x < 4^{\frac{1}{3}}$. If $x < 0$, then $x^3 < 0$ and so $x^3 < 4$. Hence $A = (-\infty, 4^{\frac{1}{3}})$. 
$\sup A = 4^{\frac{1}{3}}$ and $\inf A$ DOES NOT exist.

(5) 
\[ A = \{ x \cos x | 0 \leq x \leq \pi \} \]
Observe: \( f(0) = 0, f\left(\frac{\pi}{2}\right) = 0, f(\pi) = -\pi \). The graph may look like Fig. 3.2. By calculus, \( f \) has a max and min value on \([0, \pi]\).

\[
f'(x) = x \sin x + \cos x = 0 \Rightarrow \cos x = x \sin x, \tan x = \frac{1}{x}
\]

Then \( x \approx .87 \) and \( x \cos x \approx .56 \). So \( \sup A = .56 \) (check by graph), \( f(\pi) = -\pi \) is the minimum value and \( \inf A = -\pi \).

(6)

\[
A = \{x \mid x^2 + x - 6 < 0\}
\]

\( x^2 + x - 6 = (x + 3)(x - 2) = 0 \) when \( x = -3 \) or \( x = 2 \).

For \( x = 0 \), we can get \( x^2 + x - 6 = -6 < 0 \). So \( A = (-3, 2) \) and

\[
\sup A = 2, \quad \inf A = -3
\]

2/3/2010

Q: Is \( 2^k + 1 \) prime for any \( k \in \mathbb{N} \)?

A: No!

\[
k = 5, 2^5 + 1 = 32 + 1 = 4294967297 = 641 \cdot 6700417
\]

Examples:

(1) Let \( A \subset \mathbb{R} \) and suppose \( \sup A = \inf A \). What can we say about \( A \)?

Let \( w = \sup A = \inf A \). If \( a \in A \), then

\[
w = \sup A \Rightarrow w \geq a
\]

\[
w = \inf A \Rightarrow w \leq a
\]

So \( w = a \) and \( A = \{w\} \).

(2) Let \( A \subset \mathbb{R} \) and \( B \subset A \). What can we say about \( \sup A \) and \( \sup B \)?

Assuming both \( A \) and \( B \) are bounded. Let \( w = \sup A \), then \( w \) is an upper bound for \( A \). Since \( B \subset A \), \( w \) is also an upper bound for \( B \). Hence \( w \geq \sup B \).

**Exercise:** What can you prove about \( \inf A \) and \( \inf B \)?

(3)

\[
A = \left\{\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \ldots\right\}
\]

\[
a_n = \frac{n}{n+2} = \frac{1}{1+\frac{2}{n}} \to 1 \quad (\frac{2}{n} \to 1 \text{ when } n \to 1)
\]

or \( \frac{n}{n+2} = \frac{n+2-2}{n+2} = 1 - \frac{2}{n+2} \to 1 \).

So \( \sup A = 1 \) and \( \inf A = \frac{1}{3} \). Note that \( \sup A \) is not an element of \( A \) but \( \inf A \in A \).
The Least upper bound axiom
Math statement that the reals \( \mathbb{R} \) have no "holes". Equivalently, if we approach a number as a l.u.b, then that number exists.

**Least upper bound/complete axiom**
Every non-empty set of real numbers that is bounded above has a least upper bound.

From this, we get a version of the well-ordering theorem for the reals.

**Theorem 0.1.** Let \( A \neq \emptyset \), \( A \subset \mathbb{R} \) and \( A \) bounded below. Then \( \text{glb} A \) exists.

**Proof.** Consider \( B = \{-a | a \in A\} \). Since \( A \) is bounded below, \( \exists x \in \mathbb{R}, \forall a \in A, a \geq x \). Then \( \forall a \in A, -a \leq -x \) and \( -x \) is an upper bound for \( B \). By LUB axiom, \( B \) has a l.u.b., say \( y = \text{lub}(B) \).

Claim: \(-y = \text{glb}(A)\).

First, we want to show that \(-y\) is a lower bound.

\[ \forall a \in A, -a \leq y \Rightarrow \forall a \in A, a \geq -y \]

and \(-y\) is a lower bound.

Second, we have to show that \(-y\) is the greatest one. Suppose \(-y < r\), then \( y > -r \). Since \( y = \text{lub}(B) \), \( \exists a \in A, y > -a > -r \). Then \( a < r \) and \( r \) is not a lower bound for \( A \). So \(-y = \text{glb}(A) \). \( \square \)
An important consequence is:
The natural numbers \( \mathbb{N} \) and in fact the set \( A_r = \{ nr|n \in \mathbb{N} \} \) for any positive real \( r \) are unbounded above.

**Theorem 0.2** (Archimedean Property of Reals). Let \( a, b \) be positive real numbers, then \( \exists n \in \mathbb{N}, na > b \).

*Proof.* By contradiction. Suppose \( \forall n \in \mathbb{N}, na \leq b \). Then \( A = \{ na|n \in \mathbb{N} \} \) is bounded above. Let \( b^* = lub(A) \). Since \( a > 0 \), \( b^* - a \) is not an upper bound. So \( \exists m \in \mathbb{N}, b^* - a < ma \). This implies

\[
b = (b^* - a) + a < ma + a = (m + 1)a
\]

contradicts that \( b^* \) is an upper bound for \( A \). So \( \exists n \in \mathbb{N}, na > b \). \( \square \)

**Corollary 0.3.**
1. \( \mathbb{N} \) is unbounded above.
2. \( glb\{ \frac{1}{n} | n \in \mathbb{N} \} = 0 \)

*Proof.*
1. \( \mathbb{N} = \{ n \cdot 1 | n \in \mathbb{N} \} \) is unbounded by A.P. \( (a = 1) \).
2. For any \( r > 0 \), we want to show \( \exists n, \frac{1}{n} < r \). Since

\[
\frac{1}{n} < r \iff 1 < nr
\]

This follows from A.P. \( (b = 1, a = r) \). So 0 is the greatest lower bound. \( \square \)

**Exercises**

1. Let \( a > 0 \). Then

\[
\text{glb}\{ \frac{a}{n} | n \in \mathbb{N} \} = 0
\]

2. Prove the following variant of A.P.:

\[
\text{Let } a, b > 0, \text{ then } \exists n \in \mathbb{N}, -na < -b
\]

This means \( \{-na|n \in \mathbb{N}\} \) and \( \{-n|n \in \mathbb{N}\} \) are unbounded in **NEG SENSE** (goes to \(-\infty\)).

3. Prove: If \( a > 0 \), then \( lub\{ -\frac{a}{n} | n \in \mathbb{N} \} = 0 \)

**2/5/2010**

**Theorem 0.4.** There is a real number \( x \) such that \( x^2 = 2 \).
Proof. Let $S = \{s \in \mathbb{R} | s > 0 \text{ and } s^2 < 2 \}$. Since $1 \in S$, $S$ is not empty. Moreover, 2 is an upper bound. This can be proved by contrapositive:
If $r \geq 2$, then $r^2 \geq 2^2 = 4 > 2 \Rightarrow r$ is not in $S$.
By LUB axiom, sup $S$ exists. Let $x = \sup S > 1$.

Claim: $x^2 \geq 2$ and $x^2 \leq 2$, which says $x^2 = 2$.
Suppose $x^2 < 2$, then $b = 2 - x^2 > 0$. Set $a = 2x + 1$. By exercise (1), $\exists n \in \mathbb{N}, \frac{a}{n} < b$ i.e. $\frac{1}{n}(2x + 1) < x^2 - x^2 = x^2 < 2$.
$\Rightarrow \frac{1}{n}(2x + 1) < \frac{1}{n}(2x + 1) < 2 - x^2$.
$\Rightarrow x^2 + \frac{2}{n} + (\frac{1}{n})^2 < 2, (x + \frac{1}{n})^2 < 2$ and $x + \frac{1}{n} \in S$.
This contradicts to the fact that $x = \sup S$, so $x^2 \geq 2$.

A similar argument shows that if $x^2 < 2$, we can find $n \in \mathbb{N}$ with $(x - \frac{1}{n})^2 > 2$, contradicting that $x$ is the smallest upper bound. So we also have $2 \leq x^2$ and hence $x^2 = 2$. \(\square\)

Now we want to show that there are rational numbers everywhere.

**Theorem 0.5.** Let $a, b$ be real numbers with $0 < a < b < 1$, then $\exists r \in \mathbb{Q}$ with $a < r < b$.

*Proof.* Since $b > a, b - a > 0$. Since glb$\{\frac{a}{n} | n \in \mathbb{N}\} = 0$, we have $n_1, n_2 \in \mathbb{N}$ with $\frac{1}{n_1} < b - a$ and $\frac{1}{n_2} < a$. Let $n = n_1 n_2$, then $\frac{1}{n} < b - a$ and $\frac{1}{n} < a$ (see Fig.3.3). Let $B = \{\frac{j}{n} | 1 \leq j \leq n$ and $\frac{1}{n} \leq a\}$ $B \neq \emptyset$ since $\frac{1}{n} \in B$ and bound above by 1. By LUB axiom, $B$ has a max element $\frac{m}{n}$. (since $B$ is finite, lub$(B) \in B$). Then $\frac{m+1}{n} > a$. Also $\frac{m}{n} + \frac{1}{n} < a + (b - a) = b$. So we can choose $r = \frac{m+1}{n}$. \(\square\)

**Theorem 0.6 (n-th roots of positive numbers).** Let $n \in \mathbb{N}$ and $y > 0$. Then $\exists x > 0$ such that $x^n = y$ i.e. $x = y^{\frac{1}{n}} = \sqrt[n]{y}$.

**Examples:** find lub and glb if they exist:

1. $A = \{x | x^2 < 4\} = \{x | x < 2\} = (-2, 2)$
   lub$(A) = 2$, glb$(A) = -2$.

2. $B = \{x | x^5 > 9\}$
   $(x$ negative $\Rightarrow x^5$ negative $) \Rightarrow x > 0$.
   If $9 \leq x^5$, then $9^\frac{1}{5} \leq (x^5)^\frac{1}{5} = x$ (Why? Check it). So $B = \{x | x > 9^\frac{1}{5}\} = [9^\frac{1}{5}, +\infty)$
   No lub, glb$(B) = 9^\frac{1}{5} \in B$.

3. $C = \{2\frac{1}{2}, 2\frac{1}{4}, 2\frac{1}{3}, \cdots\} = \{2 + \frac{1}{n} | n \geq 2\}$
   lub$(C) = 2 + \frac{1}{2} \in C$.
   glb$(C) = 2 + glb\{\frac{1}{n} | n \geq 2\} = 2 + 0 = 2$ not in $C$. 

(4) \( D = \{ x | x > 0 \text{ and } \ln x < 1 \} \).
\( \ln x = 1 \Rightarrow x = e \).
So \( D = (0, e), \text{glb}(D) = 0, \text{lub}(D) = e \).

(5) \[
A = \left\{ \frac{1}{2}, -\frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{3}{4}, -\frac{1}{4}, \ldots \right\}
\]
\( \sup(A) = 1, \inf(A) = -\frac{1}{2} \).

(6) \[
A = \left\{ 0, \frac{1}{2}, -\frac{1}{2}, \frac{3}{4}, -\frac{1}{4}, \frac{7}{8}, -\frac{1}{8}, \frac{15}{16}, -\frac{1}{16}, \ldots, \frac{2^n - 1}{2^n}, -\frac{1}{2^n}, \ldots \right\}
\]
\( \sup(A) = 1, \inf(A) = -\frac{1}{2} \).
Find \( a \in A \) with \( a > .99 \):
\( n = 7, \frac{2^7 - 1}{2^7} = 1 - \frac{1}{128} > 1 - \frac{1}{100} = .99 \)
Find \( a \in A \) with \( a > .999 \):
\( n = 10, \frac{2^{10} - 1}{2^{10}} = 1 - \frac{1}{1024} > 1 - \frac{1}{1000} = .999 \)

(7) \[
A = \{ x | x^3 + x > 0 \} = \{ x | x > 0 \}
\]
\( x^3 + x = x(x^2 + 1) = 0 \Rightarrow x = 0 \)
No sup \( B \), inf \( B = 0 \) (see Fig.3.4).

(8) \[
B = \{ x | x^3 - x > 0 \}
\]
\( x^3 - x = 0 = x(x - 1)(x + 1) \)
\( B = \{ x | x > 1 \text{ or } -1 < x < 0 \} = (-1, 0) \cup (1, +\infty) \)
No sup \( B \), inf \( B = -1 \) (see Fig.3.5).