(2/8) **Sequences**

Look at exercises 1.17-1.26 in section 3.1

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**Sequences**

Some prerequisites from calculus:

**Limits**

- **Limits at a point \( a \in \mathbb{R} \)**
  - Let \( f \) be defined in an open interval about \( a \in \mathbb{R} \).

- **Limits at infinity**
  - Let \( f \) be defined on some ray \( [R, +\infty) \).

Intuitively,

- \( \lim_{x \to a} f(x) = L \) means "as \( x \) gets closer and closer to \( a \), \( f(x) \) gets closer and closer to \( L \)."

Let \( \varepsilon > 0 \) is an "error" or "tolerance" by which we allow \( f(x) \) to differ from \( L \).

Let \( \delta > 0 \), or \( N \), is the "deviation number"; it depends on \( \varepsilon \).

In terms of error,

- \( \lim_{x \to a} f(x) = L \) means that given \( \varepsilon > 0 \), we can find \( \delta > 0 \) such that if \( |x - a| < \delta \), then \( |f(x) - L| < \varepsilon \).

- \( \lim_{x \to +\infty} f(x) = L \) means that given \( \varepsilon > 0 \), we can find \( N \) such that if \( x > N \), then \( |f(x) - L| < \varepsilon \). (i.e., \( L \) is a horizontal asymptote)

**Formal definition of continuity:**

- \( f \) is continuous at \( x = a \) if, given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( |x - a| < \delta \) then \( |f(x) - f(a)| < \varepsilon \).

**Definition of a sequence**

A sequence is a function \( a : \{ n \in \mathbb{Z}^+ \mid n \geq K \} \to \mathbb{R} \) for some \( K \in \mathbb{Z}^+ \). We typically use \( K = 0 \) (domain: \( \mathbb{Z}^+ \)) or \( K = 1 \) (domain: \( \mathbb{N} \)). We usually write \( a_n = a(n) \), and we write the sequence as \( \langle a_n \rangle \) or \( \langle a_n \rangle_{n=K}^{+\infty} \). Note that the sequence \( \langle a_n \rangle \) and the set \( \{ a_n \mid n \geq K \} \) of its values are not the same thing. (E.g., if \( a_n = \{1, n \text{ even} \} \) is defined for \( n \geq 1 \), then \( \{ a_n \mid n \geq 1 \} \) is just the two-element set \( \{1, -1\} \).)
Limits of sequences

This concept is similar to the “limit at infinity” of a function. We say that 

\[ \lim_{n \to \infty} a_n = L \]

or 

\[ a_n \to L \text{ as } n \to \infty \]

or \( \langle a_n \rangle \) converges to \( L \), if 

\( a_n \) gets closer and closer to \( L \) as \( n \) gets larger and larger.

**Formal Definition**

The sequence \( \langle a_n \rangle \) converges to the limit \( L \) (written \( \lim_{n \to \infty} a_n = L \)) if for every \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that if \( n > N \), 

\[ \left| a_n - L \right| < \varepsilon \]

(“error”) ("cutoff point") then \( |a_n - L| < \varepsilon \).

Connection between Functions and Sequences

If \( F: [0, +\infty) \to \mathbb{R} \), \( \lim_{x \to +\infty} F(x) = L \), and we define a sequence \( a_n = F(n) \) (the values of \( F \) on \( \mathbb{N} \)), then \( \lim_{n \to \infty} a_n = L \). (Just compare the definitions, and “round up” the \( N \) to the nearest integer if necessary.)

This will allow us to use techniques of calculus (e.g. L'Hôpital's Rule) to investigate sequences.

Convergence/Divergence

We say that a sequence converges if \( \lim_{n \to \infty} a_n \) exists, and diverges otherwise. There are multiple ways for a sequence to diverge:

1. \( \lim_{n \to \infty} a_n = +\infty \) or \( -\infty \); (examples: \( a_n = n^2 \) (\( a_n \to +\infty \)); \( a_n = \ln(n) \) (\( a_n \to -\infty \)).

2. More than one possible limiting value; (example: \( \langle a_n \rangle = 1, -1, 1, -1, 1, -1, \ldots \)).

3. No possible limiting value; (example: \( a_n = \text{nth digit in the decimal expansion of } \pi = 3.14159\ldots \)).

Boundedness

The sequence \( \langle a_n \rangle \) is bounded if the set \{ \( a_n \mid n \in \mathbb{N} \} \) (i.e. the range of the function \( a \)) is bounded (so \( \exists R \in \mathbb{R} \) such that \( \forall n \in \mathbb{N}, \left| a_n \right| < R \)).

Theorem. If \( \langle a_n \rangle \) is convergent, then it is bounded.

Sketch of proof. Take \( \varepsilon = 1 \) in the definition of \( \lim_{n \to \infty} a_n = L \). Then \( L - 1 < a_n < L + 1 \) for all but a finite number of the \( a_n \); consider the maximum absolute value of these, and compare with \( |L - 1| \) and \( |L + 1| \).
(2/10) Last time, we proved the following: if \( \{a_n\} \) is convergent, then \( \{a_n\} \) is bounded. (The converse is not true: why?) The contrapositive — if \( \{a_n\} \) is unbounded, then \( \{a_n\} \) is divergent — gives us a test for divergence.

E.g. Let \( r > 1 \) be given. We'll show that \( \{r^n\} \) diverges by showing it's unbounded: let \( M > 0 \) be given, and find \( n \in \mathbb{N} \) such that \( r^n > M \). But \( r^n > M \Leftrightarrow n \ln r > \ln M \Leftrightarrow n > \frac{\ln M}{\ln r} \), and such an \( n \) certainly exists since \( \mathbb{N} \) is bounded above.

**Exercise.** What happens if \( 0 < r < 1 \)? If \( r = 1 \)?

The algebra of limits

**Limit Laws** Suppose \( \{a_n\}, \{b_n\} \) are sequences, with

\[
\lim_{n \to \infty} a_n = L, \quad \lim_{n \to \infty} b_n = M.
\]

(1) \( \lim_{n \to \infty} (a_n + b_n) = L + M \)

(2) If \( c \in \mathbb{R} \), \( \lim_{n \to \infty} (ca_n) = cL \)

(3) \( \lim_{n \to \infty} (a_n b_n) = LM \)

(4) If \( M > 0 \) and \( b_n > 0 \) \( \forall n \in \mathbb{N} \),

\[
\lim_{n \to \infty} \left( \frac{1}{b_n} \right) = \frac{1}{M}
\]

(5) If \( M > 0 \) and \( b_n > 0 \) \( \forall n \in \mathbb{N} \),

\[
\lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = \frac{L}{M}
\]

**Proofs**

(1), (2) and (3): see course notes.

(5) follows from (3) and (4) by writing

\[
\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n}.
\]

We prove (4): let \( \varepsilon > 0 \) be given. We need to show

\[
\exists n_0 \in \mathbb{N} \text{ such that if } n > n_0, \text{ then } \left| \frac{1}{b_n} - \frac{1}{M} \right| < \varepsilon.
\]

Now \( \frac{1}{b_n} - \frac{1}{M} = \frac{M - b_n}{M b_n} = \frac{1}{M b_n} (M - b_n) \). Since

\[
b_n \to M, \exists n_1 \in \mathbb{N} \text{ such that if } n > n_1, \text{ then } b_n > \frac{M}{2}.
\]

(Why?) So, if \( n > n_1 \),

\[
\frac{1}{b_n} < \frac{1}{M/2} = \frac{2}{M^2}.
\]

On the other hand, since \( b_n \to M \) and \( \frac{M^2}{2} \varepsilon \) is positive, \( \exists n_2 \in \mathbb{N} \) such that if \( n > n_2 \), then \( |M - b_n| < \frac{M^2}{2} \varepsilon \).

Finally, put \( n_0 = \max(n_1, n_2) \), so \( n_0 > n_1 \) and \( n_0 > n_2 \).

Whenever \( n > n_0 \),

\[
\left| \frac{1}{b_n} - \frac{1}{M} \right| = \left| \frac{1}{M b_n} \right| \left| M - b_n \right| < \frac{2}{M^2} \frac{M^2}{2} \varepsilon = \varepsilon,
\]

and so \( \frac{1}{b_n} \to \frac{1}{M} \) as claimed.

**Corollary.** If \( \lim_{n \to \infty} a_n = M \), then \( \lim_{n \to \infty} a_n - M = 0 \) and \( \lim_{n \to \infty} |a_n - M| = 0 \).

**Pinching/Squeeze Theorem** Let \( \{a_n\}, \{b_n\}, \{c_n\} \) be sequences such that, for some \( k \in \mathbb{N} \), \( a_n \leq c_n \leq b_n \) for all \( n > k \). Then if \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = L \), we must \( \lim_{n \to \infty} c_n = L \).
By using the limit laws, and the fact that \( \lim_{n \to \infty} \frac{1}{n} = 0 \), we can find limits of rational functions.

For example, we can show (easily) that

\[
\lim_{n \to \infty} \frac{5n^5 + 7n^3 + 2n + 8}{4n^5 - 9n^4 - 3n^2} = \frac{5}{4} \quad \text{and} \quad \lim_{n \to \infty} \frac{n^3 + 2n^2 + 3n}{n^3 - 3n^2 - 1} = 0.
\]

(First step: pull out the largest power from numerator and denominator)

**True or False?**

1. \( \langle a_n \rangle, \langle b_n \rangle \) divergent \( \Rightarrow \) \( \langle a_n b_n \rangle \) divergent?  
   (FALSE: take \( a_n = b_n = \frac{1}{(-1)^n} \), \( n \) odd; )

2. \( \langle a_n b_n \rangle \) divergent \( \Rightarrow \) at least one of \( \langle a_n \rangle, \langle b_n \rangle \) divergent?  
   (TRUE: contrapositive of Limit Law (3))

**E.g.** Show \( \frac{2^n}{n!} \to 0 \). (Idea: Pinch \( \frac{2^n}{n!} \) between 0 and \( a_n \), with \( a_n \to 0 \).

\[
\text{But } \frac{2^n}{n!} = 2 \cdot \frac{2}{3} \cdot \frac{2}{4} \cdots \frac{2}{n-1} \cdot \frac{2}{n} \leq \frac{2}{n}, \text{ so } a_n = \frac{2}{n} \to 0 \text{ works.)}
\]

(2/12) Given a sequence \( \langle a_n \rangle \), we can form the subsequences \( \langle e_n \rangle \), \( e_n = a_{2n} \), of even-index terms and \( \langle o_n \rangle \), \( o_n = a_{2n+1} \), of odd-index terms.

**Exercises.** Prove:

1. \( a_n \to L \), then \( e_n \to L \) and \( o_n \to L \).
2. \( a_n \to L \) and \( e_n \to M \), with \( L \neq M \), then \( \langle a_n \rangle \) diverges.

**Claim.** If \( e_n \to L \) and \( o_n \to L \), then \( a_n \to L \).

**Proof.** Let \( \varepsilon > 0 \) be given. \( \exists n_1 \in \mathbb{N} \) such that \( n > n_1 \implies |a_{2n+1} - L| < \varepsilon \), and \( \exists n_2 \in \mathbb{N} \) such that \( n > n_2 \implies |a_{2n} - L| < \varepsilon \). But \( n_0 = 2n_1n_2 \); then if \( n > n_0 \), \( |a_n - L| < \varepsilon \) whether \( n \) is even or odd, so \( a_n \to L \).

**Sequences and functions**

**Theorem.** If \( \langle a_n \rangle \) is a sequence such that \( a_n \to L \), and \( f: \mathbb{R} \to \mathbb{R} \) is a function which is continuous at \( L \), then \( f(a_n) \to f(L) \).

**Proof.** Let \( \varepsilon > 0 \). Since \( f \) is continuous at \( L \), \( \exists \delta > 0 \) such that \( |x - L| < \delta \implies |f(x) - f(L)| < \varepsilon \).

Since \( a_n \to L \), \( \exists n_0 \in \mathbb{N} \) such that \( n > n_0 \implies |a_n - L| < \delta \implies |f(a_n) - f(L)| < \varepsilon \), so in fact \( f(a_n) \to f(L) \).

This result, together with the fact that if \( \lim_{x \to \infty} f(x) = M \) and \( a_n = f(n) \) then \( a_n \to M \), will allow us to compute many limits.
Examples

1. \( \lim_{n \to \infty} \sin^2 \left( \frac{\pi}{n^3} \right) = 0. \) Why? \( \frac{1}{x^3} \to 0 \) as \( x \to \infty \), so \( \frac{1}{n^3} \to 0 \) as \( n \to \infty \), and thus\( \pi \left( \frac{1}{n^3} \right) \to 0. \) But \( \sin \left( \frac{\pi}{n^3} \right) \) is continuous at \( 0 \), so
\( \sin^2 \left( \frac{\pi}{n^3} \right) \to \sin^2 (0) = 0. \)

2. \( \lim_{n \to \infty} \cos (\ln (1 - \frac{1}{n})) = 1, \) because: \( 1 - \frac{1}{n} \to 1, \) \( \ln (1) = 0, \) \( \cos (0) = 1, \)
\( \ln \) is continuous at \( 1, \)
and \( \cos \) is continuous at \( 0. \)

3. \( \lim_{n \to \infty} (1 + \frac{1}{n})^n = e. \)

If we can show \( \lim_{x \to \infty} f(x) = e, \) where \( f(x) = \left(1 + \frac{1}{x}\right)^x, \) we've done. Consider \( \ln f(x) = x \cdot \ln \left(1 + \frac{1}{x}\right). \)
Rewrite this as \( \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x}} \), which is an \( \frac{\infty}{\infty} \) form as \( x \to \infty \); thus L'Hopital's Rule applies.
\( \lim_{x \to \infty} \ln f(x) = \lim_{x \to \infty} \frac{\ln (1 + \frac{1}{x})}{\frac{1}{x}} = \lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}} \cdot \frac{1}{x^2} = \lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}} = 1, \)
so \( \lim_{x \to \infty} f(x) = e^1 = e, \) as claimed.

(Similar reasoning shows that \( \lim_{n \to \infty} n^{1/n} = 1, \) and for any \( a \neq 0 \), \( \lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n = e^a. \))

Remark. Suppose \( \lim_{x \to 0} f(x) = L \), if \( y = f \left( \frac{1}{x} \right) \), then \( b = L \) (exercise). For example, since \( \frac{\sin x}{x} \to 1 \) as \( x \to 0, \) we have \( n \cdot \sin \left( \frac{1}{n} \right) = \frac{\sin \left( \frac{1}{n} \right)}{\frac{1}{n}} \to 1 \) as \( n \to \infty. \)

Limits at infinity

Formally, a sequence \( \langle a_n \rangle \) diverges to \( \infty \) if \( \forall M \in \mathbb{R} \ \exists N \in \mathbb{N} \) such that \( n > N \Rightarrow a_n > M, \) and diverges to \( -\infty \) if \( \forall M \in \mathbb{R} \ \exists N \in \mathbb{N} \) such that \( n > N \Rightarrow a_n < -M. \) (Intuitively, \( a_n \to \infty \) if \( a_n \) gets arbitrarily larger and larger as \( n \to \infty, \) and \( a_n \to -\infty \) if \( a_n \) gets "larger in the negative sense" as \( n \to \infty. \))

Examples Find \( \langle a_n \rangle, \langle b_n \rangle \) such that \( a_n \to \infty, b_n \to \infty, \) and \( (i) \ \frac{a_n}{b_n} \to \infty \) \( (ii) \ \frac{a_n}{b_n} \to 0 \)
\( (iii) \ \frac{a_n}{b_n} \to -\infty \) \( (iv) \ \frac{a_n}{b_n} \to -\infty. \)

(i): take \( a_n = n, \) \( b_n = n. \)

(ii): take \( a_n = n, \) \( b_n = n^2. \)

(iii, iv): impossible!

Exercises to look at: 2.11(a,b); 2.14, 2.15
in section 3.2;
3.7, 4.4, 4.5 in sections 3.3 and 3.4.