MATH3283W LECTURE NOTES: WEEK 6

2/22/2010
Recursive sequences (cont.)

Examples:

(2) $a_1 = 2,\ a_{n+1} = \frac{1}{3-a_n}$.
The first few terms are

$$2, 1, \frac{1}{2}, \frac{1}{\frac{5}{2}} = \frac{2}{5}, \frac{1}{\frac{13}{5}} = \frac{5}{13}, \cdots$$

Since $\frac{5}{13} < \frac{2}{5}$, we suspect that a_n is a decreasing sequence. Let's prove it by induction:
$a_2 < a_1$ is true.
Suppose $a_{n+1} < a_n$, then we want to show that $a_{n+2} = \frac{1}{3-a_{n+1}} < a_{n+1}$. First, we need to show that $\{a_n\}$ is bounded.
Claim: $a_n \leq 2$ (by induction).
It's true for $a_1 = 2$.
If $a_n \leq 2$, then $a_{n+1} = \frac{1}{3-a_n} \leq \frac{1}{3-2} \leq 2$. So $a_n \leq 2$ is true by induction.
Now $3 - a_{n+1} > 3 - a_n \geq 1 > 0$. So $a_{n+2} = \frac{1}{3-a_{n+1}} < \frac{1}{3-a_n} = a_{n+1}$.
We also need to claim: $0 < a_n,\ \forall n$.
True for $n = 1$.
Assume it is true for n, then $3 - a_n > 3 - 2 = 1$ and $\frac{1}{3-a_n} > 0$.

By bounded convergence theorem, $a_n \to L$ for some L. Using recursive sequences,

$$L = \lim_{n\to\infty} a_{n+1} = \frac{1}{3 - \lim_{n\to\infty} a_n} = \frac{1}{3-L}$$

$$L(3-L) = 3L - L^2 = 1$$

$$L^2 - 3L + 1 = 0 \Rightarrow L = \frac{3 \pm \sqrt{9-4}}{2}$$

So $L = \frac{3-\sqrt{5}}{2} \approx .382$. (Q: why do we know $L \neq \frac{3+\sqrt{5}}{2}$?)

Newton's Method

formula: check calculus textbook.

Examples

1 Method of finding square roots.
Consider $f(x) = x^2 - 2$. $f(x) = 0$ when $x = \pm\sqrt{2}$.
Using Newton method: guess $x_1 = \frac{3}{2}$, then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{2x_n^2 - x_n^2 + 2}{2x_n} = \frac{x_n^2 + 2}{2x_n} = \frac{1}{2}(x_n + \frac{2}{x_n}) \quad (*)$$

Rule: divide 2 by guess and average with guess (divide and average method). Does it work?
First, we show: $\forall n, x_n > \sqrt{2}$.
$x_1 = \frac{3}{2} > \sqrt{2}$ true.
Assume $x_n > \sqrt{2}$, we want to show $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n}) > \sqrt{2}$,
i.e. $\frac{x_n^2+2}{2x_n} > \sqrt{2}$, same as $x_n^2 - 2\sqrt{2}x_n + 2 > 0$.
But $x_n^2 - 2\sqrt{2}x_n + 2 = (x_n - \sqrt{2})^2 > 0$ is always true.
Second, we have to show that $\{x_n\}$ is decreasing, or

$$x_{n+1} = \frac{x_n^2 + 2}{2x_n} \leq x_n \text{ i.e. } x_n^2 + 2 \leq 2x_n^2 \text{ or } 2 \leq x_n^2$$

But this is true by (1).
To compute L, we use (*) and take limits:

$$L = \frac{L^2 + 2}{2L}, 2L^2 = L^2 + 2, L = \pm\sqrt{2}$$

Since $\forall n, x_n > \sqrt{2}$, we know $L = \sqrt{2}$.
How well does it work? $\sqrt{2} = 1.4142135$
$x_1 = \frac{3}{2} = 1.5, x_2 = \frac{1}{2}(\frac{3}{2} + \frac{4}{3}) = \frac{17}{12} = 1.4167$
$x_3 = \frac{1}{2}(\frac{17}{12} + \frac{24}{17}) = \frac{577}{408} = 1.4142056$ accurate to 5 decimal places!

2 Newton method and inversion.
To find $\frac{1}{a}$ (if $a > 0$), we need to solve

$$f(x) = \frac{1}{x} - a = 0$$

Use Newton method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\frac{1}{x_n} - a}{-\frac{1}{x_n^2}} = x_n + \frac{x_n - ax_n^2}{1} = 2x_n - ax_n^2$$

Note: replace division (inversion) by multiplication.
$a = 13, \frac{1}{13} = .076923$

$$x_1 = .08, x_2 = x_1(2 - 13x_1) = .08 \times .96 = 0.0768,$$
$$x_3 = x_2(2-13x_2) = .0768\times 1.0016 = 0.0769228 \text{ 5 place accuracy}$$

Continued fraction expansion

Consider recursive sequence $a_1 = 1,\ a_{n+1} = 1 + \frac{1}{1+a_n}$.

$$\text{even terms} \qquad \text{odd terms}$$
$$a_2 = 1 + \frac{1}{1+1} = 1\frac{1}{2}, a_3 = 1 + \frac{1}{1+\frac{3}{2}} = 1\frac{2}{5}$$
$$a_4 = 1 + \frac{1}{1+\frac{7}{5}} = 1\frac{5}{12}, a_5 = 1 + \frac{1}{1+\frac{17}{12}} = 1\frac{12}{29}$$

It turns out that $a_2 = x_1$ and $a_4 = x_2$ from Newtons method to compute $\sqrt{2}$ starting with $x_1 = \frac{3}{2}$. It can be shown that a_{2n} is decreasing, bounded below and $a_n \to \sqrt{2}$.

Also a_{2n+1} is increasing and $a_{2n+1} \to \sqrt{2}$. This means

$$\sqrt{2} = \frac{1}{1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}}$$

For example, $a_1 = 1, a_2 = 1 + \frac{1}{1+1}, a_3 = 1 + \frac{1}{1+\frac{1}{1+1}}$.

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Example: Find Cubic roots

To find $2^{\frac{1}{3}}$, solve $f(x) = x^3 - 2 = 0$.

Newton method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2}{3x_n^2} = \frac{2}{3}(\frac{x_n^3+1}{x_n^2}) = \frac{2}{3}(x_n + \frac{1}{x_n^2})$$

set $x_1 = 2$, $x_2 = \frac{2}{3}(2 + \frac{1}{4}) = \frac{2}{3} \cdot \frac{9}{4} = \frac{3}{2} < x_1$

$x_{n+1} = \frac{2}{3}x^n + \frac{2}{3x_n^2} < x_n$ if $\frac{2}{3x_n^2} < \frac{1}{3}x_n$ i.e. $x_n^3 \geq 2$

Show by induction:

$x_1^3 = 8 > 2$

$x_n^3 > 2, x_n^2 \leq 2, n \geq 3$

So $\lim x_n$ exists, $L = \frac{2}{3}(\frac{L^3+1}{L^2}), 3L^3 = 2L^3 + 2, L^3 = 2, L = 2^{\frac{1}{3}}$

values: $x_1 = 2, x_2 = \frac{3}{2}, x_3 = \frac{2}{3}(\frac{3}{2} + \frac{4}{9}) = \frac{2}{3} \cdot \frac{35}{18} = \frac{35}{27} = 1.296296$

$x_4 = \frac{2}{3}(\frac{35}{27} + \frac{729}{1225}) = \frac{70}{81} + \frac{486}{1225} = 1.26095$

$2^{\frac{1}{3}} = 1.25992$, error$\approx .001$

Exercise: Ch3 3.7, 4.4, 4.5(a)-(d), 5.5, 5.6(a)-(e), 6.6, 6.7, 6.8, 6.11

Cauchy sequences

A sequence is called Cauchy if terms get closer and closer to another as n gets larger and larger.
Formally, $< a_n >$ is Cauchy if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, (m, n \geq n_0 \Rightarrow |a_m - a_n| < \varepsilon)$$

Theorem 0.1. *If $< a_n >$ is convergent, then $< a_n >$ is Cauchy.*

Proof. Let $\varepsilon > 0$ and $L = \lim_{n\to\infty} a_n$. Since $< a_n >$ is convergent,

$$\exists n_0 \in \mathbb{N}, (n \geq n_0 \Rightarrow |a_n - L| < \frac{\varepsilon}{2})$$

If $m, n \geq n_0$, then

$$|a_m - a_n| = |(a_m - L) + (L - a_n)| \leq |a_m - L| + |L - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

We now investigate whether Cauchy sequences converge.

Theorem 0.2. *Let $< a_n >$ Cauchy. Then $< a_n >$ is bounded.*

Proof. Let $\varepsilon = 1$. $< a_n >$ Cauchy implies $\exists n_0 \in \mathbb{N}, s.t. |a_m - a_n| < 1$ if $m, n \geq n_0$. In particular, for any $m \geq n_0, |a_m - a_{n_0}| < 1$ and

$$|a_m| = |a_m - a_{n_0} + a_{n_0}| \leq |a_m - a_{n_0}| + |a_{n_0}| < 1 + |a_{n_0}|$$

A bound for $|a_1|, |a_2|, \cdots, |a_{n_0-1}|$ is

$$M = \max\{|a_1|, |a_2|, \cdots, |a_{n_0-1}|\}$$

So $n \in \mathbb{N} \Rightarrow |a_n| < M + |a_{n_0}| + 1$ □

Corollary 0.3. *Let $< a_n >$ be Cauchy, then*

$$< a_n > \text{ is monotone } \Rightarrow < a_n > \text{ converges.}$$

Proof. Since $< a_n >$ is bounded, if $< a_n >$ is monotone, then $< a_n >$ converges by the bounded convergent theorem. □

A very hard result is:
If $< a_n >$ is a sequence, then $< a_n >$ contains another sequence $< b_m >$ (a subsequence $< a_{n_m} >$, like odd or even terms), which is monotone.
Now let $< a_n >$ Cauchy, and $< b_m >$ is a monotone subsequence of $< a_n >$.

$$< a_n > \text{ is bounded } \Rightarrow < b_m > \text{ is bounded}$$

So $< b_m >$ converges to some limit L. Since $< a_n >$ is Cauchy, the terms b_m "attract" the terms a_n and $a_n \to L$.

Find monotone subsequences of Cauchy sequences:
a_m is called a peak of point if it satisfies ($a_m > a_n$ if $n > m$). Thus a_m is a strict upper bound for $\{a_n | n > m\}$.
Let $< a_n >$ be a sequence. Then there are two cases:
(1) Suppose $< a_n >$ has infinitely many peak points $a_{n_1} < a_{n_2} < \cdots < a_{n_k} < \cdots$. Then $< a_{n_k} >$ is a (strictly) decreasing subsequence.
(2) Suppose $< a_n >$ has only finitely many peak points $a_{n_1}, a_{n_2}, \cdots, a_{n_k}$ (It may have none: $a_n = 1 - \frac{1}{n}, n \geq 1$), then $a_{n_1} > a_{n_2} > \cdots > a_{n_k}$. Let $m_1 = n_k + 1$.
Since a_{m_1} is not a peak point, $\exists m_2 > m_1$ s.t. $a_{m_2} \geq a_{m_1}$.
Since a_{m_2} is not a peak point, $\exists m_3 > m_2$ s.t. $a_{m_3} \geq a_{m_2}$.
continue: if $a_{m_j} \geq a_{m_{j-1}}$, then a_{m_j} is not a peak point.
$\Rightarrow \exists m_{j+1} > m_j$ with $a_{m_{j+1}} \geq a_{m_j}$
Thus, we can construct a subsequence $< a_{m_j} >$. Moreover, $< a_{m_j} >$ is increasing.

2/26/2010

Summation Notation

Consider a sequence $a_1, a_2, \cdots < a_n, \cdots$. We will look at ways to add these terms. First need nice ways to add them. Suppose $m < n$. Then sigma notation for adding is

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$$

i is a "dummy" variable. Thus

$$\sum_{j=m}^{n} a_j = \sum_{k=m}^{n} a_k = \sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$$

Examples
1. $\sum_{k=m}^{n} a_k = \sum_{k=0}^{n-m} a_{k+m}$
2. $\sum_{j=m}^{2m} a_j + \sum_{i=2m+1}^{3m} a_i = \sum_{k=m}^{3m} a_k$
3. Write $2^4 + 3^4 + \cdots + n^4$ in sigma notation.

$$\sum_{j=2}^{n} j^4 = \sum_{j=0}^{n-2} (j+2)^4 = \sum_{j=n+2}^{2n} (j-n)^4$$

Examples

(1) Let $s_n = \sum_{i=1}^{n} \frac{3}{n}[(\frac{i}{n})^2 + 1]$. Determine if $\lim_{n\to\infty} s_n$ exists.

$$\begin{aligned} s_n &= \frac{3}{n}[\sum_{i=1}^{n}(\frac{i}{n})^2 + \sum_{i=1}^{n} 1] \\ &= \frac{3}{n}\sum_{i=1}^{n}\frac{i^2}{n^2} + \frac{3}{n}\cdot n \\ &= \frac{3}{n^3}(\sum_{i=1}^{n} i^2) + 3 \\ &= \frac{3}{n^3}\frac{n(n+1)(2n+1)}{6} + 3 \\ &= \frac{6n^3 + 9n^2 + 3n}{6n^3} + 3 \to 1 + 3 = 4 \end{aligned}$$

(2) What is the value of $\sum_{j=0}^{n}(-1)^j$?

$$\begin{aligned} \sum_{j=0}^{n}(-1)^j &= (-1)^0 + (-1)^1 + \cdots + (-1)^n \\ &= 1 - 1 + 1 - 1 + \cdots + (-1)^n \\ &= \begin{cases} 1 & \text{if n is even} \\ 0 & \text{if n is odd} \end{cases} \end{aligned}$$

(3) Expand $\sum_{i=1}^{n}(5^i - 5^{i-1})$

$$\sum_{i=1}^{n}(5^i - 5^{i-1}) = \sum_{i=1}^{n} 5^i - \sum_{i=1}^{n} 5^{i-1} = \sum_{i=1}^{n} 5^i - \sum_{i=0}^{n-1} 5^i = 5^n - 5^0 = 5^n - 1$$

Theorem 0.4 (Generalized triangle inequality)**.** *Let $n \in \mathbb{N}$ and $a_1, a_2, \cdots, a_n$ real. Then*

$$|\sum_{i=1}^{n} a_i| \le \sum_{i=1}^{n} |a_i|$$

Proof. Induction. True for $n = 1$:

$$|\sum_{i=1}^{1} a_i| = |a_1| = |\sum_{i=1}^{1} |a_i|$$

Assume true for n. Then

$$\begin{aligned} |\sum_{i=1}^{n+1} a_i| &= |\sum_{i=1}^{n} a_i + a_{n+1}| \leq |\sum_{i=1}^{n} a_i| + |a_{n+1}| \quad \text{(usual triangle inequality)} \\ &\leq \sum_{i=1}^{n} |a_i| + |a_{n+1}| = \sum_{i=1}^{n+1} |a_i| \quad \text{(induction)} \end{aligned}$$

So it is true for all n by induction. □

Examples

(1) Assume $x \neq 1$. Show: for $n \geq 0$,

$$\sum_{k=1}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$$

$n = 0, 1 = \frac{1-x}{1-x}$ true.
Assume true for n. Then

$$\sum_{k=1}^{n+1} x^k = \sum_{k=1}^{n} x^k + x^{n+1} = \frac{1 - x^{n+1} + x^{n+1} - x^{n+2}}{1 - x} = \frac{1 - x^{n+2}}{1 - x} = \frac{1 - x^{(n+1)+1}}{1 - x}$$

(2) Simplify the following:
   (a) $\sum_{i=1}^{n}(\sum_{j=1}^{n}(i + j))$ and $\sum_{j=1}^{n}(\sum_{i=1}^{n}(i + j))$

$$\begin{aligned} \sum_{i=1}^{n}(\sum_{j=1}^{n}(i + j)) &= \sum_{i=1}^{n}(ni + \sum_{j=1}^{n} j) = \sum_{i=1}^{n} ni + n\sum_{j=1}^{n} j \\ &= n\sum_{i=1}^{n} i + n\sum_{j=1}^{n} j \\ &= 2n \cdot \frac{n(n+1)}{2} = n^2(n+1)(= \sum_{j=1}^{n}(\sum_{i=1}^{n}(i + j)).) \end{aligned}$$

   (b) $\sum_{n=1}^{i} \frac{(-1)^{n+1}5^{n-1}}{(n+1)^2 4^{n+2}}$
   $n + 1$ appear several times. so we may want to write in terms of $n + 1$:
   $5^{n-1} = 5^{n+1-2}, 4^{n+2} = 4^{(n+1)+1}$, so the series looks like

$$\sum_{n=1}^{i} \frac{(-1)^{n+1}}{(n+1)^2} \cdot \frac{5^{(n+1)-2}}{4^{(n+1)+1}}$$

Now replace $n+1=j$. Then j goes from 2 to $i+1$ and

$$\sum_{n=1}^{i}\frac{(-1)^{n+1}}{(n+1)^2}\cdot\frac{5^{(n+1)-2}}{4^{(n+1)+1}}=\sum_{j=2}^{i+1}\frac{(-1)^j}{j^2}\cdot\frac{5^{j-2}}{4^{j+1}}=\sum_{j=2}^{i+1}(-\frac{5}{4})^j\cdot\frac{1}{100j^2}$$

(c) $\sum_{i=1}^{n}\sum_{j=i}^{n}(i+j)$

Observe:

$$\sum_{j=1}^{n}(1+j)+\sum_{j=2}^{n}(2+j)+\sum_{j=3}^{n}(3+j)+\cdots=n+\sum_{j=1}^{n}j+2(n-1)+\sum_{j=2}^{n}j+3(n-2)+\sum_{j=3}^{n}j+\cdots$$