1a) Pf: Let \( B = \{ n \in \mathbb{N} \mid n \geq k \} \). We want to show that \( A = B \).

By assumption, \( A = \mathbb{N} \) & \( k \) is the smallest element of \( A \).

It follows that \( A = B \). We'll prove that \( A = B \) by contradiction. Suppose \( A \neq B \). Then the set \( B \setminus A \neq \emptyset \).

(By definition \( B \setminus A = \{ x \in B \mid x \notin A \} \).

We are guaranteed that \( B \setminus A \) has a smallest element by the well-ordering of \( \mathbb{N} \). Let \( p \) be the smallest element of \( B \setminus A \), so \( p \leq \exists q \in B \setminus A \).

Now \( k \in B \setminus A \Rightarrow p > k \), & so \( p > 1 > k \). Since \( k \leq p-1 < p \) we see that \( p-1 \in B \), but \( p-1 \notin B \setminus A \), thus \( p-1 \in A \).

Finally, assumption 2 tells us that \( n \in A \Rightarrow n+1 \in A \).

Since \( p-1 \in A \), it follows that \( (p-1)+1 = p \in A \). But \( p \in B \setminus A \Rightarrow p \notin A \). This is a contradiction. \( \therefore B \setminus A = \emptyset \), i.e. \( A = B \).

1b) Pf: Let \( A = \{ n \in \mathbb{N} \mid n \geq k \& P(n) \) is true \} \). Notice that

\[ \forall n \geq k, \; n \in A \Rightarrow P(n) \) is true \). Thus to prove \[ \forall n \geq k, \; P(n) \) is true \), it suffices to show that \[ A = \{ n \in \mathbb{N} \mid n \geq k \} \). By assumption 1, we have that \( P(k) \) is true \& so \( k \in A \). Moreover since \( \forall n \in A \), \( n \geq k \) we see that \( k \) is the smallest element of \( A \).

Assumption 2 tells us that if \( P(n) \) is true then \( P(n+1) \) is true. In terms of \( A \) this gives us that \( \forall n \in A \Rightarrow n+1 \in A \). In summary, \( A \in \mathbb{N} \& 1. \; k \) is the smallest element of \( A \) & 2. \( \forall n \in A \Rightarrow n+1 \in A \). By problem 1a) we conclude \( A = \{ n \in \mathbb{N} \mid n \geq k \} \), as desired.
2a) If: Base Case: Let \( n=1 \). Then \( 1^3 = \left( \frac{1(1+1)}{2} \right)^2 \).

Inductive step: Suppose for some \( n \in \mathbb{N} \) we have

\[
1^3 + 2^3 + \ldots + n^3 = \left( \frac{n(n+1)}{2} \right)^2,
\]

Then we have

\[
1^3 + 2^3 + \ldots + n^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3
\]

\[
= \frac{n^2(n+1)^2 + 4(n+1)^3}{4}
\]

\[
= \frac{(n+1)^2(n^2 + 4(n+1))}{4}
\]

\[
= \frac{(n+1)^2(n^2 + 4n + 4)}{4}
\]

\[
= \frac{(n+1)^2(n+2)^2}{4}
\]

\[
= \left( \frac{(n+1)(n+2)}{2} \right)^2.
\]

Thus if the formula is true for \( n \) then it is true for \( n+1 \). By induction, the formula is true for all \( n \in \mathbb{N} \).
2b) Let \( g(n) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n+1)} \).

Let's compute \( g(n) \) for the first several \( n \):

\[
\begin{align*}
\text{n=1 : } & \quad g(1) = \frac{1}{1 \cdot 2} = \frac{1}{2} \\
\text{n=2 : } & \quad g(2) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{4}{6} = \frac{2}{3} \\
\text{n=3 : } & \quad g(3) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{3}{4} \\
\end{align*}
\]

It seems reasonable to guess that \( g(n) = \frac{n}{n+1} \).

If this is the case, then \( g(n) = 1 - f(n) \Rightarrow f(n) = 1 - \frac{n}{n+1} = \frac{1}{n+1} \).

Let's prove this result with induction:

Base case: Let \( n = 1 \). Then \( \frac{1}{1 \cdot 2} = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{1+1} \).

Inductive step: Say for some \( n \in \mathbb{N} \) that

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}.
\]

Then

\[
\begin{align*}
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} & = 1 - \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
& = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
& = \frac{n(n+2) + 1}{(n+1)(n+2)} \\
& = \frac{n^2 + 2n + 1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)}
\end{align*}
\]
2b) cont,

Continuing the computation:

\[
\frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)}
\]

\[= \frac{(n+1)^2}{(n+1)(n+2)}\]

\[= \frac{n+1}{n+2}\]

\[= 1 - \frac{1}{n+2}\]

Thus \(f(n+1) = \frac{1}{n+2}\), as desired.

2c) Ans: We have

\[2^3 + 4^3 + 6^3 + \cdots + (2n)^3\]

\[= 2^3 \cdot 1^3 + 2^3 \cdot 2^3 + 2^3 \cdot 3^3 + \cdots + 2^3 \cdot n^3\]

\[= 8 \left( 1^3 + 2^3 + 3^3 + \cdots + n^3 \right)\]

\[= 8 \left( \frac{n(n+1)}{2} \right)^2 \quad [\text{by } 2a]\]

\[= 2n^2(n+1)^2.\]