1. a) Claim: \( \forall n \geq 2, \ 3^n - 2^n \geq 2^n \).

Proof: When \( n = 2 \) we have \( 3^2 - 2^2 = 5 \geq 4 = 2^2 \).

Now assume for some \( n \geq 2 \) that \( 3^n - 2^n \geq 2^n \).

Then \( 3^{n+1} - 2^{n+1} = 3 \cdot 3^n - 2 \cdot 2^n \)
\[ \geq 2(3^n - 2^n) \]
\[ \geq 2 \cdot 2^n = 2^{n+1}, \]

Thus the claim is true by induction.

Well \( 3^n - 2^n \geq 2^n \implies \frac{1}{3^n - 2^n} \leq \frac{1}{2^n} \).

We can see that \( \sum_{n=2}^{\infty} \frac{1}{2^n} = \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n \) is a convergent geometric series since \( |\frac{1}{2}| < 1 \). Since \( \forall n \geq 2 \)
\[ 0 \leq \frac{1}{3^n - 2^n} \leq \frac{1}{2^n} \), we have that \( \sum_{n=2}^{\infty} \frac{1}{3^n - 2^n} \) is convergent

by the comparison theorem.

It follows that \( \sum_{n=2}^{\infty} \frac{1}{3^n - 2^n} \) is convergent since

\[ \sum_{n=2}^{\infty} \frac{1}{3^n - 2^n} = 1 + \sum_{n=2}^{\infty} \frac{1}{3^n - 2^n}. \]

b) Observe that \( \lim_{n \to \infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2 - \sqrt{n}}} = \lim_{n \to \infty} \frac{n^2 - \sqrt{n}}{n^2} = \lim_{n \to \infty} 1 - \frac{\sqrt{n}}{n^2} = 1 > 0. \)

By the limit comparison test we have that \( \sum_{n=2}^{\infty} \frac{1}{n^2 - \sqrt{n}} \) is convergent \( \iff \sum_{n=2}^{\infty} \frac{1}{n^2} \) is convergent.

Indeed, \( \sum_{n=2}^{\infty} \frac{1}{n^2} \) converged by the \( p \)-test, so \( \sum_{n=2}^{\infty} \frac{1}{n^2 - \sqrt{n}} \) is convergent.
1. c) We will show that \( \sum_{n=1}^{\infty} \frac{2n+1}{n^{4}+1} \) is **divergent** using the limit comparison test. Observe that

\[
\lim_{n \to \infty} \frac{\frac{1}{n}}{\frac{n^{4}+1}{n^{4}+1}} = \lim_{n \to \infty} \frac{\frac{n^{4}+1}{n^{4}+1}}{2n^{2}+n} = \lim_{n \to \infty} \frac{n^{4}+1}{n^{4}} \frac{n^{4}+1}{2n^{2}+n} = \frac{1}{2} > 0.
\]

Since \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent, it follows that \( \sum_{n=1}^{\infty} \frac{2n+1}{n^{4}+1} \) is divergent.

2. For both (i) & (ii) the answer is "not necessarily." Consider the sequences defined by \( a_n = \begin{cases} \frac{1}{n^2} & n \text{ even} \\ \frac{1}{n} & n \text{ odd} \end{cases} \) & \( b_n = \frac{1}{n^2} \).

Then \( 0 < a_{2n} \leq b_n \forall n \), & \( \sum_{n=1}^{\infty} a_n \) is divergent while \( \sum_{n=1}^{\infty} b_n \) is convergent.

This provides an immediate counter-example to answering (i) & (ii) with "yes."
3. We can solve the integral first: \( \forall n \in \mathbb{N} \)

\[
\int_{n}^{2n} \frac{1}{x^{\frac{1}{3}}} \, dx = -\frac{3}{4} x^{-\frac{2}{3}} \bigg|_{n}^{2n} = -\frac{3}{4} \left( \frac{1}{(2n)^{\frac{2}{3}} - n^{\frac{2}{3}}} \right)
\]

\[
= -\frac{3}{4} \left( \frac{1}{(2^{\frac{2}{3}} - 1) n^{\frac{2}{3}}} \right) \frac{1}{n^{\frac{2}{3}}}
\]

Now, let \( c = -\frac{3}{4(2^{\frac{2}{3}} - 1)} \). (Note that \( c \) is just a constant).

We have now that

\[
\sum_{n=1}^{\infty} \int_{n}^{2n} \frac{1}{x^{\frac{1}{3}}} \, dx = \sum_{n=1}^{\infty} c \cdot \frac{1}{n^{\frac{2}{3}}}
\]

By the \( p \)-test, \( \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}} \) is convergent thus \( \sum_{n=1}^{\infty} c \cdot \frac{1}{n^{\frac{2}{3}}} \) is convergent by theorem 4.2.4(i).
4. a) **Proof:** We have that \( a_n > 0 \) & \( b_n > 0 \) for every \( n \), so we can apply the limit comparison test. Notice that

\[
\lim_{n \to \infty} \frac{a_n b_n}{a_n} = \lim_{n \to \infty} b_n = 0
\]

since \( \sum b_n \) is a convergent series. Since \( \sum a_n \) is convergent it follows that \( \sum a_n b_n \) is convergent by Theorem 4.76 (2).

b) **Claim:** If \( \sum_{n=1}^{\infty} a_n \) is a positive series which converges then \( \forall \) \( m \in \mathbb{N} \) \( \sum_{n=1}^{\infty} a_n^m \) is a convergent series.

**Proof of claim:** (by induction).

Let \( m = 1 \). Then \( \sum_{n=1}^{\infty} a_n^1 = \sum_{n=1}^{\infty} a_n \) is a convergent series by assumption.

Now assume for some \( m \in \mathbb{N} \) that \( \sum_{n=1}^{\infty} a_n^m \) is a convergent series. Since \( \forall n \in \mathbb{N} \) \( a_n > 0 \), we have that \( \sum_{n=1}^{\infty} a_n^m \) is a positive convergent series. By part (a) above we have that since \( \sum_{n=1}^{\infty} a_n^m \) & \( \sum_{n=1}^{\infty} a_n \) are positive convergent series, the series

\[
\sum_{n=1}^{\infty} a_n^m \cdot a_n = \sum_{n=1}^{\infty} a_n^{m+1}
\]

is also convergent.

By induction, we have shown that \( \sum_{n=1}^{\infty} a_n^m \) is convergent for every \( m \in \mathbb{N} \), as desired.