MATH3283W PROFESSIONAL PROBLEM 4 SOLUTIONS

(1) Let $a_n = \frac{1}{n^2 b^{2n}}(x-a)^n$. By ratio test,

$$|\frac{a_{n+1}}{a_n}| = |\frac{\frac{1}{(n+1)^2 b^{2(n+1)}}(x-a)^{n+1}}{\frac{1}{n^2 b^{2n}}(x-a)^n}| = |\frac{n^2 b^{2n}(x-a)^{n+1}}{(n+1)^2 b^{2n+2}(x-a)^n}| = |(\frac{n}{n+1})^2 \frac{x-a}{b^2}| \to |\frac{x-a}{b^2}|$$

$\because |\frac{x-a}{b^2}| < 1 \Leftrightarrow |x-a| < b^2$ and $|\frac{x-a}{b^2}| > 1 \Leftrightarrow |x-a| > b^2$.
$\therefore \sum a_n$ converges when $|x-a| < b^2$ and diverges when $|x-a| > b^2$. The radius of converges(ROC) is b^2.

$|x-a| < b^2 \Leftrightarrow -b^2 < x-a < b^2 \Leftrightarrow a-b^2 < x < a+b^2$

When $x = a-b^2$ or $x-a = -b^2$, $\sum \frac{(-b^2)^n}{n^2 b^{2n}} = \sum \frac{(-1)^n}{n^2}$ converges by alternating series test.
When $x = a+b^2$ or $x-a = b^2$, $\sum \frac{(b^2)^n}{n^2 b^{2n}} = \sum \frac{1}{n^2}$ converges by p-test. So the interval of convergence is $[a-b^2, a+b^2]$.

(2) (a) We know that the PS of $h(y) = e^y$ at $y = 0$ is $1 + \sum_{n=1}^{\infty} \frac{y^n}{n!}$ and it converges for any $y \in \mathbb{R}$. Replace y by $x^2 + x^3$, we get $h(x^2+x^3) = e^{x^2+x^3} = 1 + \sum_{n=1}^{\infty} \frac{(x^2+x^3)^n}{n!}$ and it converges for any $x \in \mathbb{R}$ (since $x^2 + x^3 \in \mathbb{R}$ as well). Consider the first 4 terms of the series,

$$\begin{aligned}
& 1 + \sum_{n=1}^{\infty} \frac{(x^2+x^3)^n}{n!} \\
= \; & 1 + \frac{x^2+x^3}{1!} + \frac{(x^2+x^3)^2}{2!} + \frac{(x^2+x^3)^3}{3!} + \sum_{n=4}^{\infty} \frac{(x^2+x^3)^n}{n!} \\
= \; & 1 + x^2 + x^3 + \frac{x^4}{2} + x^5 + \frac{x^6}{2} + \frac{x^6 + 3x^7 + 3x^8 + x^9}{6} + \sum_{n=4}^{\infty} \frac{(x^2+x^3)^n}{n!} \\
= \; & 1 + x^2 + x^3 + \frac{x^4}{2} + x^5 + \frac{2x^6}{3} + \frac{x^7}{2} + \frac{x^8}{2} + \frac{x^9}{6} + \sum_{n=4}^{\infty} \frac{(x^2+x^3)^n}{n!}
\end{aligned}$$

Since alll the terms in the second part have degree ≥ 8, the first 7 terms of the PS is $1 + x^2 + x^3 + \frac{x^4}{2} + x^5 + \frac{2x^6}{3} + \frac{x^7}{2}$.

(b) Since $(e^{x^2+x^3})' = (2x + 3x^2)e^{x^2+x^3} = g(x)$ and the PS of $e^{x^2+x^3}$ has ROC=∞ by part (a), the PS of $g(x)$ also has

ROC=∞.

(c) If $f(x) = a_0 + a_1x + \cdots + a_nx^n$ is a polynomial, then

$$f^{(k)}(x) = a_k \cdot k(k-1)\cdots 2\cdot 1 + (k+1)k\cdots 3\cdot 2x + \cdots + n(n-1)\cdots(n-k+1)x^{n-k}$$

when $1 \le k \le n$ and $f^{(k)}(x) = 0$ when $k > n$. So $f(0) = a_0$ and $f^{(k)}(0) = a_k \cdot k!$ when $1 \le k \le n$. Thus the PS of $f(x)$ is

$$f(0) + \sum_{k=1}^{\infty} \frac{f^{(k)}(0)x^k}{k!} = a_0 + \sum_{k=1}^{n} a_kx^k = f(x).$$

In particular, when $f(x) = 3x^{17} - 9x^{11} + x^7 - 5x^4 + x - 3$, the PS is also $3x^{17} - 9x^{11} + x^7 - 5x^4 + x - 3$.

(3) (a) If $x \ge 0$, $|x| = x$ and $(x^2)^{\frac{1}{2}} = x$.
If $x < 0$, $|x| = -x$ and $(x^2)^{\frac{1}{2}} = -x$.
So $|x| = (x^2)^{\frac{1}{2}}$ for any $x \in \mathbb{R}$.

(b) We know that the PS of $\cos y$ at $y = 0$ is $1 + \sum_{n=1}^{\infty} \frac{(-1)^n y^{2n}}{n!}$ and it converges for any $y \in \mathbb{R}$. If we replace y by $\sqrt{|x|^2}$, we get

$$\cos\sqrt{|x|^2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n(\sqrt{|x|^2})^{2n}}{n!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n|x|^n}{n!}$$

It converges for any $x \in \mathbb{R}$.
Note: this series actually is not a "power series" since its n-th term contains $|x|^n$ rather than x^n.

(4) A. since $\sum a_nx^n$ converges at $x = -\frac{3}{2}$, it converges absolutely for $|x| < |-\frac{3}{2}| = \frac{3}{2}$. So (i) $\sum a_n = \sum a_n \cdot 1^n$ and (ii) $\sum a_n(\frac{5}{4})^n$ both converges because $|1| < \frac{3}{2}$ and $|\frac{5}{4}| < \frac{3}{2}$.
We don't know if $\sum a_n(-3)^n$ is convergent or not. For example, $\sum \frac{x^n}{3^n}$ and $\sum \frac{x^n}{n3^n}$ both satisfy the given conditions, but $\sum \frac{x^n}{3^n}$ diverges at $x = -3$ and $\sum \frac{x^n}{n3^n}$ converges at $x = -3$.

B. Since $\sum a_nx^n$ converges when $|x| < \frac{3}{2}$, r =ROC should be at least $\frac{3}{2}$. If not($r < \frac{3}{2}$), then $r' = \frac{r+\frac{3}{2}}{2} = r + \frac{\frac{3}{2}-r}{2} > r$ and $\sum a_nx^n$ diverges at $x = r'$, which is a contradiction because $\frac{3}{2} > r'$.
If $r > 3$, then $\sum a_nx^n$ is convergent at $x = 3 > r$, a contradiction.
So we know that $\frac{3}{2} \le r \le 3$. Moreover, $\sum \frac{2^nx^n}{n3^n}$ and $\sum \frac{x^n}{n3^n}$ both satisfy the given conditions and they have ROC=$\frac{3}{2}$ and 3 respectively. So $\frac{3}{2}$ is the greatest lower bound and 3 is the

least upper bound, i.e the codition $\frac{3}{2} \leq r \leq 3$ can't be improved further.

C.

$$|a_n(-2)^n| < (\frac{4}{5})^n \Leftrightarrow |a_n| < \frac{(\frac{4}{5})^n}{|(-2)^n|} = (\frac{2}{5})^n$$

So

$$\sum_{n=100}^{\infty} |a_n x^n| \leq \sum_{n=100}^{\infty} (\frac{2}{5})^n |x^n| = \sum_{n=100}^{\infty} |\frac{2x}{5}|^n$$

and $\sum_{n=100}^{\infty} |\frac{2x}{5}|^n$ converges if $|\frac{2x}{5}| < 1$ or $|x| < \frac{5}{2}$. Hence $\sum_{n=100}^{\infty} a_n x^n$ (and $\sum_{n=1}^{\infty} a_n x^n$) converges absolutely when $|x| < \frac{5}{2}$ and the ROC is at least $\frac{5}{2}$. So the condition in B should be $\frac{5}{2} \leq r \leq 3$. It can't be improved because the series $\sum_{n=1}^{\infty} a_n x^n$ with $a_n = \frac{2^n}{n5^n}$ satisfies all the required conditions and has ROC$= \frac{5}{2}$.