(1) Let \( a_n = \frac{1}{n^2 \pi n} (x - a)^n \). By ratio test,

\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{1}{(n+1)^2 \pi n} (x - a)^{n+1}}{\frac{1}{n^2 \pi n} (x - a)^n} = \frac{n^2 b^n (x - a)^{n+1}}{(n+1)^2 b^{n+2} (x - a)^n} = \left| \frac{n}{n+1} \right|^2 \frac{x - a}{b^2} \to \left| \frac{x - a}{b^2} \right|
\]

\( \therefore \frac{x - a}{b^2} < 1 \iff |x - a| < b^2 \) and \( \frac{x - a}{b^2} > 1 \iff |x - a| > b^2 \).

\( \therefore \sum a_n \) converges when \( |x - a| < b^2 \) and diverges when \( |x - a| > b^2 \). The radius of convergence (ROC) is \( b^2 \).

When \( x = a - b^2 \) or \( x - a = -b^2 \), \( \sum \frac{(b^2)^n}{n^2 \pi n} = \sum \frac{1}{n^2} \) converges by alternating series test.

When \( x = a + b^2 \) or \( x - a = b^2 \), \( \sum \frac{(b^2)^n}{n^2 \pi n} = \sum \frac{1}{n^2} \) converges by p-test. So the interval of convergence is \( [a - b^2, a + b^2] \).

(2) (a) We know that the PS of \( h(y) = e^y \) at \( y = 0 \) is \( 1 + \sum_{n=1}^{\infty} \frac{y^n}{n!} \) and it converges for any \( y \in \mathbb{R} \). Replace \( y \) by \( x^2 + x^3 \), we get \( h(x^2 + x^3) = e^{x^2 + x^3} = 1 + \sum_{n=1}^{\infty} \frac{(x^2 + x^3)^n}{n!} \) and it converges for any \( x \in \mathbb{R} \) (since \( x^2 + x^3 \in \mathbb{R} \) as well). Consider the first 4 terms of the series,

\[
1 + \sum_{n=1}^{\infty} \frac{(x^2 + x^3)^n}{n!}
\]

\[
= 1 + \frac{x^2 + x^3}{1!} + \frac{(x^2 + x^3)^2}{2!} + \frac{(x^2 + x^3)^3}{3!} + \sum_{n=4}^{\infty} \frac{(x^2 + x^3)^n}{n!}
\]

\[
= 1 + x^2 + x^3 + \frac{x^4}{2} + x^5 + \frac{x^6}{2} + \frac{x^7}{2} + \frac{x^8}{2} + \frac{x^9}{2} + \sum_{n=4}^{\infty} \frac{(x^2 + x^3)^n}{n!}
\]

Since all the terms in the second part have degree \( \geq 8 \), the first 7 terms of the PS is \( 1 + x^2 + x^3 + \frac{x^4}{2} + x^5 + \frac{2x^6}{3} + \frac{x^8}{2} + \frac{x^9}{6} \).

(b) Since \( (e^{x^2 + x^3})' = (2x + 3x^2)e^{x^2 + x^3} = g(x) \) and the PS of \( e^{x^2 + x^3} \) has ROC=\( \infty \) by part (a), the PS of \( g(x) \) also has
ROC=∞.

(c) If \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \) is a polynomial, then
\[
f^{(k)}(x) = a_k k(k-1) \cdots 2 \cdot 1 + (k+1) k \cdots 3 \cdot 2 x + \cdots + n(n-1) \cdots (n-k+1) x^{n-k}
\]
when \( 1 \leq k \leq n \) and \( f^{(k)}(x) = 0 \) when \( k > n \). So \( f(0) = a_0 \) and
\[
f^{(k)}(0) = a_k \cdot k! \text{ when } 1 \leq k \leq n.
\]
Thus the PS of \( f(x) \) is
\[
f(0) + \sum_{k=1}^{\infty} \frac{f^{(k)}(0) x^k}{k!} = a_0 + \sum_{k=1}^{n} a_k x^k = f(x).
\]
In particular, when \( f(x) = 3x^{17} - 9x^{11} + x^7 - 5x^4 + x - 3 \), the PS is also \( 3x^{17} - 9x^{11} + x^7 - 5x^4 + x - 3 \).

(3) (a) If \( x \geq 0 \), \( |x| = x \) and \( (x^2)^{\frac{1}{2}} = x \).
If \( x < 0 \), \( |x| = -x \) and \( (x^2)^{\frac{1}{2}} = -x \).
So \( |x| = (x^2)^{\frac{1}{2}} \) for any \( x \in \mathbb{R} \).

(b) We know that the PS of \( \cos y \) at \( y = 0 \) is \( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n y^{2n}}{n!} \) and it converges for any \( y \in \mathbb{R} \). If we replace \( y \) by \( \sqrt{|x|^2} \), we get
\[
\cos \sqrt{|x|^2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (\sqrt{|x|^2})^{2n}}{n!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n |x|^n}{n!}
\]
It converges for any \( x \in \mathbb{R} \).

Note: this series actually is not a "power series" since its \( n \)-th term contains \( |x|^n \) rather than \( x^n \).

(4) A. since \( \sum a_n x^n \) converges at \( x = -\frac{3}{2} \), it converges absolutely for \( |x| < \left| -\frac{3}{2} \right| = \frac{3}{2} \). So (i) \( \sum a_n = \sum a_n \cdot 1^n \) and (ii) \( \sum a_n (\frac{5}{4})^n \) both converges because \( |1| < \frac{3}{2} \) and \( \left| \frac{5}{4} \right| < \frac{3}{2} \).
We don’t know if \( \sum a_n (-3)^n \) is convergent or not. For example, \( \sum \frac{x^n}{3^n} \) and \( \sum \frac{x^n}{n3^n} \) both satisfy the given conditions, but \( \sum \frac{x^n}{3^n} \) diverges at \( x = -3 \) and \( \sum \frac{x^n}{n3^n} \) converges at \( x = -3 \).

B. Since \( \sum a_n x^n \) converges when \( |x| < \frac{3}{2} \), \( r = \text{ROC} \) should be at least \( \frac{3}{2} \). If not \( r < \frac{3}{2} \), then \( r' = \frac{r + \frac{3}{2}}{2} = r + \frac{3-r}{2} > r \) and \( \sum a_n x^n \) diverges at \( x = r' \), which is a contradiction because \( \frac{3}{2} > r' \).
If \( r > 3 \), then \( \sum a_n x^n \) is convergent at \( x = 3 > r \), a contradiction.
So we know that \( \frac{3}{2} \leq r \leq 3 \). Moreover, \( \sum \frac{2^n x^n}{n3^n} \) and \( \sum \frac{x^n}{n3^n} \) both satisfy the given conditions and they have \( \text{ROC} = \frac{3}{2} \) and 3 respectively. So \( \frac{3}{2} \) is the greatest lower bound and 3 is the
least upper bound, i.e the condition $\frac{3}{2} \leq r \leq 3$ can’t be improved further.

C.

\[ |a_n(-2)^n| < \left(\frac{4}{5}\right)^n \iff |a_n| < \frac{\left(\frac{4}{5}\right)^n}{|(-2)^n|} = \left(\frac{2}{5}\right)^n \]

So

\[ \sum_{n=100}^{\infty} |a_n x^n| \leq \sum_{n=100}^{\infty} \left(\frac{2}{5}\right)^n |x^n| = \sum_{n=100}^{\infty} \left|\frac{2x}{5}\right|^n \]

and $\sum_{n=100}^{\infty} \left|\frac{2x}{5}\right|^n$ converges if $\left|\frac{2x}{5}\right| < 1$ or $|x| < \frac{5}{2}$. Hence $\sum_{n=100}^{\infty} a_n x^n$ (and $\sum_{n=1}^{\infty} a_n x^n$) converges absolutely when $|x| < \frac{5}{2}$ and the ROC is at least $\frac{5}{2}$. So the condition in B should be $\frac{5}{2} \leq r \leq 3$. It can’t be improved because the series $\sum_{n=1}^{\infty} a_n x^n$ with $a_n = \frac{2^n}{n5^n}$ satisfies all the required conditions and has $\text{ROC} = \frac{5}{2}$. 